Hamiltonian dynamics and spectral theory for spin-oscillators

Alvaro Pelayo, San Vu Ngoc

Hamiltonian dynamics and spectral theory for spin-oscillators

Álvaro Pelayo* and San Vũ Ngọc†

Abstract

We study the Hamiltonian dynamics and spectral theory of spin-oscillators. Because of their rich structure, spin-oscillators display fairly general properties of integrable systems with two degrees of freedom. Spin-oscillators have infinitely many transversally elliptic singularities, exactly one elliptic-elliptic singularity and one focus-focus singularity. The most interesting dynamical features of integrable systems, and in particular of spin-oscillators, are encoded in their singularities. In the first part of the paper we study the symplectic dynamics around the focus-focus singularity. In the second part of the paper we quantize the coupled spin-oscillators systems and study their spectral theory. The paper combines techniques from semiclassical analysis with differential geometric methods.

1 Introduction

Coupled spin-oscillators are 4-dimensional integrable Hamiltonian systems with two degrees of freedom constructed by "coupling" the classical spin on the 2-sphere S^2 (see Figure 3.1) with the classical harmonic oscillator on the Euclidean plane $\mathbb{R}^2$. Coupled spin-oscillators are one of the most fundamental examples of integrable systems; their dynamical behavior is rich and represents some fairly general properties of low dimensional integrable systems. The goal of this paper is to study coupled spin-oscillators from the point of view of classical and quantum mechanics, using methods from classical and semiclassical analysis.

A 4-dimensional integrable system with two degrees of freedom consists of a connected symplectic 4-manifold equipped with two almost everywhere linearly independent smooth functions which Poisson commute, i.e. two smooth functions on the manifold such that one of them is invariant along the flow of the Hamiltonian vector field generated by of the other. The most interesting geometric and dynamical features of integrable systems are encoded in their singularities, i.e the points where Hamiltonian vector fields generated by the functions are linearly dependent. Around the regular points, the dynamics is simple, and described by the Arnold-Liouville-Mineur action-angle theorem. As we will see, the dynamics near the singularities is in general much more complicated and depends heavily on the type of singularity.

Let us explain the construction of coupled spin-oscillators more precisely. Let S^2 be the unit sphere in $\mathbb{R}^3$ with coordinates $(x,\, y,\, z)$, and let $\mathbb{R}^2$ be equipped with coordinates $(u,\, v)$. Let $\lambda, \rho > 0$ be positive constants. Let M be the product manifold $S^2 \times \mathbb{R}^2$ equipped with the product symplectic structure $\lambda\omega_{S^2} \oplus \rho\omega_0$. Let $J,\, H \colon M \to \mathbb{R}$ be the smooth maps defined by $J := \rho(u^2+v^2)/2 + \lambda z$ and $H := \frac{1}{2}\,(ux+vy)$. A *coupled spin-oscillator* is a 4-dimensional integrable system of the form $(M,\, \lambda\omega_{S^2} \oplus \rho\omega_0,\, (J,\, H))$, where ω_{S^2} is the standard symplectic form on the sphere and ω_0 is the standard symplectic form on $\mathbb{R}^2$.

The singularities of coupled spin-oscillators are non-degenerate and of elliptic-elliptic, transversally-elliptic (both of these types are usually referred to as "elliptic singularities") or focus-focus type. They have

*Partially supported by an NSF Postdoctoral Fellowship.
†Partially supported by an ANR 'Programme Blanc'.

infinitely many transversally-elliptic singularities (along a piecewise smooth curve, as we shall see), one elliptic-elliptic singularity at $(0, 0, -1, 0, 0)$ and one singularity of focus-focus type at $(0, 0, 1, 0, 0)$. The J component of this system is the Hamiltonian (or momentum map) of the S^1-action that simultaneously rotates about the vertical axes of the 2-sphere, and about the origin of $\mathbb{R}^2$. The H component is given as follows. Using the natural embedding of S^2 in $\mathbb{R}^3$, let π_z be the orthogonal projection from S^2 onto $\mathbb{R}^2$ viewed as the $z = 0$ hyperplane. Let $(x, y, z) \in S^2$ and $(u, v) \in \mathbb{R}^2$. Under the flow of J the points (x, y, z) and (u, v) are moving along the flows of z and $(u^2 + v^2)/2$, respectively, with the same angular velocity. Hence the inner product $\langle \pi_z(x, y, z), (u, v)\rangle = ux + vy = 2H$ is constant and commutes with J.

Because H does not come from an S^1-action, coupled spin-oscillators are not toric integrable systems – they are what now is called *semitoric integrable systems*, or simply *semitoric systems*. Semitoric systems form a rich class of integrable systems, commonly found in simple physical models. For simplicity, throughout this paper we assume the rescaling $\lambda = \rho = 1$. The statements and proofs extend immediately to the case of $\lambda, \rho > 0$, but we feel that the notation is already sufficiently heavy so we shall avoid carrying these parameters.

Semitoric integrable systems

Our interest in semitoric integrable systems was motivated by the remarkable convexity results for Hamiltonian torus actions by Atiyah [1], Guillemin-Sternberg [14], and Delzant [6]. Despite important contributions by Arnold, Duistermaat [8], Eliasson [10], Vũ Ngọc [23, 25], Zung [29] and many others, the singularity theory of integrable systems from the point of view of symplectic geometry is far from being completely understood. As a matter of fact, very few integrable systems are understood. The singularities of these systems encode a vast amount of information about the symplectic dynamics and geometry of the system, much of which is not computable with the current methods.

This singularity theory is interesting not only from the point of view of semiclassical analysis and symplectic geometry, but it also shares many common features with the study of singularities in the context of symplectic topology [20, 16], algebraic geometry and mirror symmetry (see [13] and the references therein).

The coupled spin-oscillator is perhaps the simplest non-compact example of an integrable system of semitoric type. Precisely, a *semitoric integrable system* on M is an integrable system $J, H \in \mathrm{C}^\infty(M, \mathbb{R})$ for which the component J is a proper momentum map for a Hamiltonian circle action on M and the map $F := (J, H) : M \to \mathbb{R}^2$ has only non-degenerate singularities in the sense of Williamson [27], without real-hyperbolic blocks. This means that in addition to the well-known elliptic singularities of toric systems, semitoric systems may have *focus-focus singularities*.

Semitoric integrable systems on 4-manifolds have been symplectically classified by the authors in [18, 19] in terms a collection of five invariants. While conceptually they are more easily describable, some of these invariants are involved to compute explicitly for a particular integrable system. The most difficult invariant to compute is the so called Taylor series invariant, which classifies a neighborhood of the *focus-focus singular fiber* of F. This invariant, which was introduced in [23], encodes a large amount of information about the local and semiglobal behavior of the system. Focus-focus singular fibers are singular fibers that contain some fixed point m (i.e. $\mathrm{rank}(\mathrm{d}F) = 0$) which is of *focus-focus* type, meaning that there are symplectic coordinates locally near m in which $m = (0, 0, 0, 0)$, $\omega = \mathrm{d}\xi \wedge \mathrm{d}x + \mathrm{d}\eta \wedge \mathrm{d}y$ and $F = F(m) + (x\xi + y\eta, x\eta - y\xi) + \mathcal{O}((x, \xi, y, \eta)^3)$.

Dynamics and singularities of coupled spin-oscillators

The coupled spin-oscillator system has non-degenerate singularities of elliptic-elliptic, transversally-elliptic and focus-focus type. It has exactly one singularity of focus-focus type. Near the focus-focus singularity, the behavior of the Hamiltonian vector fields generated by the system is not 2π-periodic, as it occurs with toric systems.

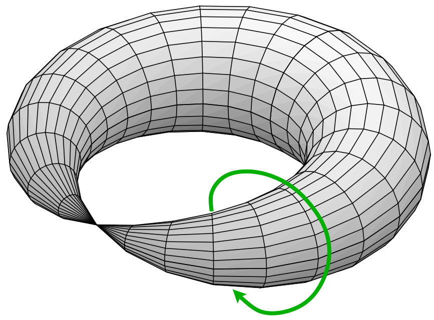

Figure 1.1: Singularity of focus-focus type and vanishing cycle. Topologically a fiber containing a single focus-focus singularity is a pinched torus.

Loosely speaking, one of the components of the system is indeed 2π-periodic, but the other one generates an arbitrary flow which turns indefinitely around the focus-focus singularity and which, as F tends to the critical value $F(m)$, deviates from periodic behavior in a logarithmic fashion, up to a certain error term; this deviation from being logarithmic is a symplectic invariant and can be made explicit – it is in fact given by an infinite Taylor series $(S)^{\infty}$ on two variables X, Y with vanishing constant term. This was proven by the second author in [23]. The goal of the first part of the present paper is compute the linear approximation of this deviation.

Theorem 1.1. *The coupled spin–oscillator is a semitoric integrable system, with one single focus-focus singularity at $m = (0, 0, 1, 0, 0) \in S^2 \times \mathbb{R}^2$. The semiglobal dynamics around m may be described as follows: the linear deviation from exhibiting logarithmic behavior in a saturated neighborhood of m is given by the linear map $L\colon \mathbb{R}^2 \to \mathbb{R}$ with expression $L(X, Y) = \frac{\pi}{2} X + 5 \ln 2\, Y$. In other words, we have an equality $(S(X, Y))^{\infty} = L(X, Y) + \mathcal{O}(X, Y)^2$, where $(S(X, Y))^{\infty}$ denotes the Taylor series invariant at the focus-focus singularity.*

As far as we know, this theorem gives the first rigorous estimate in the literature of the logarithmic deviation, and hence the first explicit quantization of the symplectic dynamics around the singularity; we prove it in Section 2. The proof is computational but rather subtle, and it combines a number of theorems from integrable systems and semiclassical analysis. The method of proof of Theorem 1.1 (given in several steps) provides a fairly general algorithm to implement in the case of other semitoric integrable systems. Moreover, it seems plausible to expect that the techniques we introduce generalize to compute higher order approximations, but not immediately – indeed, the linear approximation relies on various semiclassical formulas that are not readily available for higher order approximations. In this paper we will also find the other invariants that characterize the coupled spin-oscillator (Section 3): the polygon and height invariants; these are easier to find.

Spectral theory for quantum coupled spin-oscillators

Sections 4, 5 of this paper are devoted to the spectral theory of quantum coupled spin-oscillators. The following theorem describes the quantum spin-oscillator. For any $\hbar > 0$ such that $2 = \hbar(n+1)$, for some non-negative integer $n \in \mathbb{N}$, let $\mathcal{H}$ denote the standard $n+1$-dimensional Hilbert space quantizing the sphere S^2 (see Section 4.1).

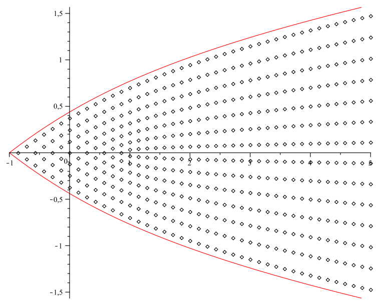

Figure 1.2: Semiclassical joint spectrum of $\hat{J}, \hat{H}$. We will explain this figure in more detail in Section 4.

Theorem 1.2. *Let $S^2 \times \mathbb{R}^2$ be the coupled spin-oscillator, and (as above) let $J,\ H \colon M \to \mathbb{R}$ be the Poisson commuting smooth functions that define it. The unbounded operators $\hat{J} := \mathrm{Id} \otimes \Big(-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}\,u^2} + \frac{u^2}{2} \Big) + (\hat{z} \otimes \mathrm{Id})$ and $\hat{H} = \frac{1}{2}(\hat{x} \otimes u + \hat{y} \otimes (\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial u})$ on the Hilbert space $\mathcal{H} \otimes \mathrm{L}^2(\mathbb{R}) \subset \mathrm{L}^2(\mathbb{R}^2) \otimes \mathrm{L}^2(\mathbb{R})$ are self-adjoint and commute. The spectrum of $\hat{J}$ is discrete and consists of eigenvalues in $\hbar(\frac{1-n}{2} + \mathbb{N})$.*

For a fixed eigenvalue λ of $\hat{J}$, let $\mathcal{E}_\lambda := \ker(\hat{J} - \lambda \mathrm{Id})$ be the eigenspace of the operator $\hat{J}$ over λ. There exists a basis $\mathcal{B}_\lambda$ of $\mathcal{E}_\lambda$ in which $\hat{H}$ restricted to $\mathcal{E}_\lambda$ is given by

$$\mathrm{M}_{\mathcal{B}_\lambda}(\hat{H}) = \left(\frac{\hbar}{2}\right)^{\frac{3}{2}} \begin{pmatrix} 0 & \beta_1 & \dots & & & 0 \\ \beta_1 & 0 & \beta_2 & & & 0 \\ 0 & \beta_2 & 0 & \beta_3 & & 0 \\ & & & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & & & & & \beta_\mu \\ 0 & 0 & \dots & & \beta_\mu & 0 \end{pmatrix},$$

where $0 \leq k \leq n$, $\ell_0 := \frac{\lambda}{\hbar} + \frac{n-1}{2}$, $\mu := \min(\ell_0, n)$, $\beta_k := \sqrt{(\ell_0 + 1 - k)k(n-k+1)}$.

The dimension of $\mathcal{E}_\lambda$ is $\mu + 1$.

Finding out how information from quantum completely integrable systems leads to information about classical systems is a fascinating "inverse" problem with very few precise results at this time. Section 5 explains how information of the coupled spin-oscillator, including its *linear* singularity theory (computed in Section 2), may be recovered from the quantum semiclassical spectrum.

The way in which we recover this linear singularity theory relies on a conjecture for Toeplitz operators, which has been proven for pseudodifferential operators. We explain in detail how to do this and formulate the following conjecture about semitoric integrable systems: that a semitoric system is determined up to symplectic equivalence by its semiclassical joint spectrum, i.e. the set of points in $\mathbb{R}^2$ where on the x-axis we have the eigenvalues of $\hat{J}$, and on the vertical axis the eigenvalues of $\hat{H}$ restricted to the λ-eigenspace of $\hat{J}$. From any such spectrum one can construct explicitly the associated semitoric system. We give strong evidence of this conjecture for the coupled spin oscillators.

Acknowledgements. The work on this article started during a short but intense visit of the second author to Berkeley. He is grateful to the Berkeley maths department, and in particular to Alan Weinstein and Maciej Zworski for their invitation. Part of this paper was written while the first author was a Professeur Invité in the Équations aux Dérivées Partielles Section at the Institut de Recherches Mathématiques de Rennes (Université Rennes 1) during January 2010, and he thanks them for the warm hospitality. He also thanks MSRI for hospitality during the Fall of 2009 and Winter 2010 when we has a member, and the University of Paris-Orsay for their hospitality during the author's visit on February 2010, during which a portion of this paper was written.

2 Singularity theory for coupled spin-oscillators

This section considers semiglobal properties. It is independent of Section 3 which concerns global properties. The main goal of this section is to prove Theorem 1.1.

Let $(M,\, \omega,\, F := (J,\, H))$ be a semitoric integrable system. Recall that a *singular point*, or a *singularity*, is a point $p \in M$ such that $\mathrm{rank}(\mathrm{d}F)(p) < 2$, where $F := (J,\, H) \colon M \to \mathbb{R}^2$. A *singular fiber* of the system is a fiber of $F \colon M \to \mathbb{R}^2$ that contains some singular point.

Let m be a focus-focus singular point m. Let $B := F(M)$. Let $\tilde{c} = F(m)$. The set of regular values of F is $\mathrm{Int}(B) \setminus \{\tilde{c}\}$, the boundary of B consists of all images of elliptic singularities, and the fibers of F are connected (see [25]).

We assume that the critical fiber $\mathcal{F}_m := F^{-1}(\tilde{c})$ contains only one critical point m, which according to Zung [28] is a generic condition, and let $\mathcal{F}$ denote the associated singular foliation.

By Eliasson's theorem [10] there exist symplectic coordinates $(x_1,\, x_2,\, \xi_1,\, \xi_2)$ in a neighborhood U around m in which $(q_1,\, q_2)$, given by

$$q_1 = x_1\xi_2 - x_2\xi_1,\ \ q_2 = x_1\xi_1 + x_2\xi_2, \tag{2.1}$$

is a momentum map for the foliation $\mathcal{F}$ (in the sense that for some local diffeomorphism $q = g \circ F$, so the maps q and F have the same fibers); here the critical point m corresponds to coordinates $(0,\, 0,\, 0,\, 0)$. Because of the uniqueness of the S^1-action one may chose Eliasson's coordinates [22] such that $q_1 = J$.

2.1 Construction of the singularity invariant at a focus-focus singularity

Fix $A' \in \mathcal{F}_m \cap (U \setminus \{m\})$ and let Σ denote a small 2-dimensional surface transversal to $\mathcal{F}$ at the point A', and let Ω be the open neighborhood of $\mathcal{F}_m$ which consists of the leaves which intersect the surface Σ.

Since the Liouville foliation in a small neighborhood of Σ is regular for both F and $q = (q_1,\, q_2)$, there is a local diffeomorphism φ of $\mathbb{R}^2$ such that $q = \varphi \circ F$, and we can define a global momentum map $\Phi = \varphi \circ F$ for the foliation, which agrees with q on U. Write $\Phi := (H_1,\, H_2)$ and $\Lambda_c := \Phi^{-1}(c)$. For simplicity we write $\Phi = q$. Note that $\Lambda_0 = \mathcal{F}_m$. It follows from (2.1) that near m the H_1-orbits must be periodic of primitive period 2π.

Suppose that $A \in \Lambda_c$ for some regular value c. Let $\tau_2(c) > 0$ be the time it takes the Hamiltonian flow associated with H_2 leaving from A to meet the Hamiltonian flow associated with H_1 which passes through A, and let $\tau_1(c) \in \mathbb{R}/2\pi\mathbb{Z}$ the time that it takes to go from this intersection point back to A, hence closing the trajectory. We denote by γ_c the corresponding loop in Λ_c.

Write $c = (c_1,\, c_2) = c_1 + \mathrm{i}c_2$, and let $\ln z$ for a fixed determination of the logarithmic function on the complex plane. Let

$$\left\{ \begin{array}{rcl} \sigma_1(c) & = & \tau_1(c) - \Im(\ln c) \\ \sigma_2(c) & = & \tau_2(c) + \Re(\ln c), \end{array} \right.$$

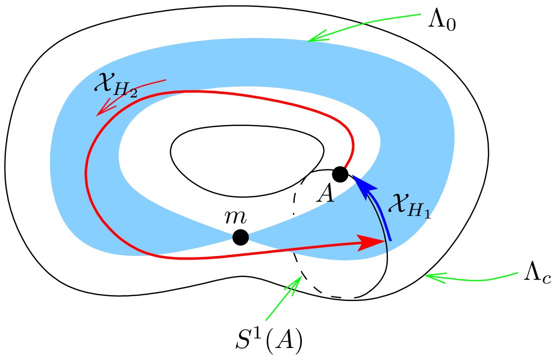

Figure 2.1: Singular foliation near the leaf $\mathcal{F}_m$, where $S^1(A)$ denotes the S^1-orbit generated by $H_1 = J$.

where $\Re$ and $\Im$ respectively stand for the real an imaginary parts of a complex number. Vũ Ngọc proved in [23, Prop. 3.1] that σ_1 and σ_2 extend to smooth and single-valued functions in a neighbourhood of 0 and that the differential 1-form

$$\sigma := \sigma_1 \,\mathrm{d}c_1 + \sigma_2 \,\mathrm{d}c_2$$

is closed. Notice that if follows from the smoothness of σ_2 that one may choose the lift of τ_2 to $\mathbb{R}$ such that $\sigma_2(0) \in [0,\, 2\pi)$. This is the convention used throughout. Following [23, Def. 3.1] , let S be the unique smooth function defined around $0 \in \mathbb{R}^2$ such that

$$\mathrm{d}S = \sigma, \quad S(0) = 0.$$

The Taylor expansion of S at $(0,\, 0)$ is denoted by $(S)^\infty$.

The Taylor expansion $(S)^\infty$ is a formal power series in two variables with vanishing constant term, and we say that $(S)^\infty$ is the *Taylor series invariant of* $(M,\, \omega,\, (J,\, H))$ *at the focus-focus point* c.

2.2 The coupled spin-oscillators

Let S^2 be the unit sphere in $\mathbb{R}^3$ with coordinates $(x,\, y,\, z)$, and let $\mathbb{R}^2$ be equipped with coordinates $(u,\, v)$. Recall from the introduction that the coupled-spin oscillator model is the product $S^2 \times \mathbb{R}^2$ equipped with the product symplectic structure $\omega_{S^2} \oplus \omega_0$ given by $\mathrm{d}\theta \wedge \mathrm{d}z \oplus \mathrm{d}u \wedge \mathrm{d}v$, and with the smooth Poisson commuting maps $J,\, H \colon M \to \mathbb{R}$ given by $J := (u^2 + v^2)/2 + z$ and $H := \frac{1}{2}(ux + vy)$. Sometimes we denote the coupled spin-oscillator by the triple $(S^2 \times \mathbb{R}^2,\, \omega_{S^2} \oplus \omega_0,\, (J,\, H))$. A simple verification leads to the following observation.

Proposition 2.1. *The coupled spin–oscillator* $(S^2 \times \mathbb{R}^2,\, \omega_{S^2} \oplus \omega_0,\, (J,\, H))$ *is a completely integrable system, meaning that the Poisson bracket* $\{J,\, H\}$ *vanishes everywhere*[1].

In addition, the map J *is the momentum map for the Hamiltonian circle action of* S^1 *on* $S^2 \times \mathbb{R}^2$ *that rotates simultaneously horizontally about the vertical axes on* S^2*, and about the origin on* $\mathbb{R}^2$*.*

The singularities of the coupled spin–oscillator are non-degenerate and of elliptic-elliptic, transversally-elliptic or focus-focus type. It has exactly one focus-focus singularity at the "North Pole" $((0,\, 0,\, 1),\, (0,\, 0)) \in S^2 \times \mathbb{R}^2$ *and one elliptic-elliptic singularity at the "South Pole"* $((0,\, 0,\, -1),\, (0,\, 0))$.

[1] equivalently the Hamiltonian vector field $\mathcal{X}_J$ is constant along the flow of $\mathcal{X}_H$

Corollary 2.2. *The coupled spin–oscillator* $(S^2 \times \mathbb{R}^2,\, \omega_{S^2} \oplus \omega_0,\, (J,\, H))$ *is a semitoric integrable system.*

Computing the Taylor series invariant at the focus-focus singularity is rather involved. At this point we are able to compute the first two terms $a_1,\ a_2$ (for the coupled spin-oscillators). Even in this case one has to do a delicate coordinate analysis of flows involving Eliasson's coordinates, and the computation of various integrals.

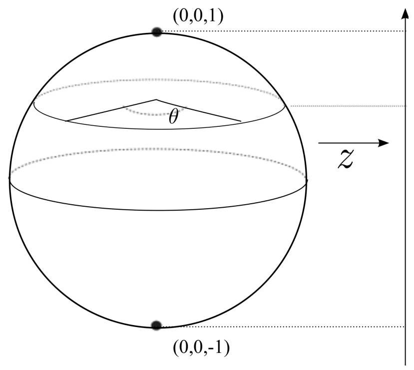

Figure 2.2: Spin model with momentum map z. Here $(\theta,\, z)$ are the angle-height coordinates on the unit sphere S^2.

2.3 Set up for coupled spin-oscillators — Integral formulas for singularity invariant

Throughout we let $M = S^2 \times \mathbb{R}^2$ and $F = (J,\, H)$. In this set up stage we introduce the 1-forms $\kappa_{1,c}$ and $\kappa_{2,c}$ in terms of which the Taylor series in defined in [23], and we recall limit integral formulas for the Taylor series invariant. Then we introduce the limit theorem proved in the semiclassical paper [22, Proposition 6.8], which will be the key ingredient for the computation .

The formulas that we present here do not correspond to the exact statements in the corresponding papers, but can be immediately deduced from it assuming the context of the present paper.

The one forms $\kappa_{1,c}$ and $\kappa_{2,c}$. As usual, we denote by $\mathcal{X}_{q_i}$ the Hamiltonian vector field generated by q_i, $i = 1,\, 2$. Let c be a fixed regular value of F. Let $\kappa_{1,c} \in \Omega^1(\Lambda_c),\ \kappa_{1,c} \in \Omega^1(\Lambda_c)$ be the smooth 1-forms on the fiber $\Lambda_c := F^{-1}(c)$ corresponding to the value c defined by the conditions

$$\kappa_{1,c}(\mathcal{X}_{q_1}) := -1, \quad \kappa_{1,c}(\mathcal{X}_{q_2}) := 0, \tag{2.2}$$

and

$$\kappa_{2,c}(\mathcal{X}_{q_1}) := 0, \quad \kappa_{2,c}(\mathcal{X}_{q_2}) := -1. \tag{2.3}$$

Note that the conditions in (2.2) and (2.3) are enough to determine $\kappa_{1,c}$ and $\kappa_{2,c}$ on Λ_c because $\mathcal{X}_{q_1},\ \mathcal{X}_{q_2}$ form a basis of each tangent space.

We will call $\kappa_{1,0}$, $\kappa_{2,0}$ the corresponding form defined in the same way as $\kappa_{1,c}$, $\kappa_{2,c}$, but only on $\Lambda_0 \backslash \{m\}$, where $m = (0, 0, 1, 0, 0)$ is the singular point of the focus-focus singular fiber Λ_0.

Remark 2.3 The forms $\kappa_{1,c},\ \kappa_{2,c}$, $i = 1,\, 2$ are closed. See also [22, Section 3.2.1]. $\oslash$

Limit integral formula for Taylor invariants. The following result will be key for our purposes in the present paper.

Lemma 2.4. *Let $(S) \in \mathbb{R}[[X, Y]]$ be the Taylor series invariant of the coupled-spin oscillator. Then the first terms of the Taylor series are given by the limits of integrals $a_1 = \lim_{c\to 0}\Big(\int_{\gamma_c}\kappa_{1,c} + \arg(c)\Big)$ and $a_2 = \lim_{c\to 0}\Big(\int_{\gamma_c}\kappa_{2,c} + \ln|c|\Big)$.*

Proof. It follows from the definition of the dynamical invariants $\tau_1(c)$ and $\tau_2(c)$ in Section 2 and the definition of $\kappa_{1,c}$ and $\kappa_{2,c}$ in (2.2) and (2.3) respectively that $\tau_i(c) = \int_{\gamma_c}\kappa_{i,c},\ \ i = 1, 2$. The first two terms of the Taylor series invariant $\sigma_1(0)$ and $\sigma_2(0)$ where $\sigma_1 = \tau_1 + \arg(c)$ and $\sigma_2 = \tau_2 - \ln|c|$.

Since σ_1 and σ_2 are smooth, we have that $a_1 = \sigma_1(0) = \lim_{c\to 0}\Big(\int_{\gamma_c}\kappa_{1,c} + \arg(c)\Big)$ and $a_2 = \sigma_2(0) = \lim_{c\to 0}\Big(\int_{\gamma_c}\kappa_{2,c} + \ln|c|\Big)$. □

Localization on the critical fiber. On the other hand, we have the following [22, Proposition 6.8] result proved by the second author.

Theorem 2.5 ([22])**.** *Let γ_0 be a radial simple loop. The integrals in Lemma 2.4 are respectively equal to*

$$a_1 = \lim_{c\to 0}\Big(\int_{\gamma_c}\kappa_{1,c} + \arg(c)\Big) = \lim_{(s,\,t)\to(0,\,0)}\Big(\int_{A_0=\gamma_0(s)}^{B_0=\gamma_0(1-t)}\kappa_{1,0} + (t_A - \theta_B)\Big), \tag{2.4}$$

and

$$a_2 := \lim_{c\to 0}\Big(\int_{\gamma_c}\kappa_{2,c} + \ln|c|\Big) = \lim_{(s,\,t)\to(0,\,0)}\Big(\int_{A_0:=\gamma_0(s)}^{B_0(t):=\gamma_0(1-t)}\kappa_{2,0} + \ln(r_{A_0}\rho_{B_0})\Big), \tag{2.5}$$

where for any point A in M close to m with Eliasson coordinates $(x_1,\, x_2,\, \xi_1,\, \xi_2)$ as defined in equation (2.1), we denote by $(r_A,\, t_A,\, \rho_A,\, \theta_A)$ the polar symplectic coordinates[2] of A, i.e. $(r_A,\, t_A)$ are polar coordinates corresponding to $(x_1,\, x_2)$ and $(\rho_A,\, \theta_A)$ are polar coordinates corresponding to $(\xi_1,\, \xi_2)$.

2.4 Computation of integral limit formulas for coupled spin-oscillators

Now, in order to apply Theorem 2.5 we need to find the curve γ_0, as well as the 1-form κ and the coordinates $(r,\, \theta,\, \rho,\, \alpha)$, both of which are defined on Λ_0. First we describe a parametrization of Λ_0, and then we use this parametrization to define γ_0. We have divided the computation into five steps.

Stage 1 — Eliasson's coordinates $(x_1,\, x_2,\, \xi_1,\, \xi_2)$

We find explicitly symplectic coordinates $(\hat{x}_1,\, \hat{x}_2,\, \hat{\xi}_1,\, \hat{\xi}_2) \in M = S^2 \times \mathbb{R}^2$ in which the "momentum map" $F\colon M \to \mathbb{R}^2$ for the coupled spin-oscillator has the form (2.1), up to a third order approximation, i.e. up to $(\mathcal{O}(\hat{x}_1,\, \hat{x}_2,\, \hat{\xi}_1,\, \hat{\xi}_2))^3$. For brevity write $\mathcal{O}(3) = (\mathcal{O}(\hat{x}_1,\, \hat{x}_2,\, \hat{\xi}_1,\, \hat{\xi}_2))^3$.

[2] These coordinates $(r_A,\, t_A,\, \rho_A,\, \theta_A)$ should not be confused with the coordinates $(r,\, t,\, \rho,\, \theta)$ without the subscript, which are coordinates in $\mathbb{R}^2 \times S^2$.

Lemma 2.6. *Consider the map* $\hat{\phi}\colon \mathrm{T}_{(0,0,0,0)}\mathbb{R}^4 \to \mathrm{T}_{(0,0,1,0,0)}(S^2\times\mathbb{R}^2)$ *given by*

$$\phi(\hat{x}_1,\, \hat{x}_2,\, \hat{\xi}_1,\, \hat{\xi}_2) = (v := \frac{1}{\sqrt{2}}(\hat{x}_2+\hat{\xi}_1),\ x := \frac{1}{\sqrt{2}}(\hat{x}_2-\hat{\xi}_1),\ u := \frac{1}{\sqrt{2}}(-\hat{x}_1+\hat{\xi}_2),\ y := \frac{1}{\sqrt{2}}(\hat{x}_1+\hat{\xi}_2)).$$

The map $\hat{\phi}$ *is a linear symplectomorphism, i.e. an automorphism such that* $\phi^*\Omega = \omega_0$*, where* $\omega_0 = \mathrm{d}\hat{x}_1 \wedge \mathrm{d}\hat{\xi}_1 \oplus \mathrm{d}\hat{x}_2 \wedge \mathrm{d}\hat{\xi}_2$ *is the standard symplectic form on* $\mathbb{R}^4$*, and* $\Omega = (\omega_{S^2} \oplus \mathrm{d}u\wedge\mathrm{d}v)_{\restriction T_{(0,0,1,0,0)}(S^2\times\mathbb{R}^2)}$ *(recall* ω_{S^2} *is the standard symplectic form on* S^2*). In addition,* $\hat{\phi}$ *satisfies the equation* $\mathrm{Hess}(\widetilde{F})\circ\hat{\phi} = (q_1,\, q_2)$*, where* $\widetilde{F} := B\circ(F-F(m)) = B\circ(F-(1,\,0)) : M \to \mathbb{R}^2$*, for the matrix* $B := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

In the above statement, we identify a Hessian with its associated quadratic form on the tangent space.

Stage 2 — Curve and Singular Fiber Parametrization

Parametrization of Λ_0. Let's now parametrize the singular fiber $\Lambda_0 := F^{-1}(1,\,0)$, where $F = (J,\,H)$ as usual. This singular fiber Λ_0 corresponds to the system of equations $J = 1$ and $H = 0$, which explicitly is given by system of two nonlinear equations $J = (u^2+v^2)/2 + z = 0$ and $H = \frac{1}{2}(ux+vy) = 0$. on the coordinates $(x,\,y,\,z,\,u,\,v)$ on the coupled spin oscillator $M = S^2\times\mathbb{R}^2$.

In order to solve this system of equations we introduce polar coordinates $u + \mathrm{i}v = r\,\mathrm{e}^{\mathrm{i}t}$ and $x+\mathrm{i}y = \rho\mathrm{e}^{\mathrm{i}\theta}$ where recall that the 2-sphere $S^2 \subset \mathbb{R}^3$ is equipped with coordinates $(x,\,y,\,z)$, and $\mathbb{R}^2$ is equipped with coordinates $(u,\,v)$.

For $\epsilon = \pm 1$, we consider the mapping $S_\epsilon : [-1,\,1]\times\mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2\times S^2$ given by the formula $S_\epsilon(p) = (r(p)\,\mathrm{e}^{\mathrm{i}t(p)},\, (\rho(p)\,\mathrm{e}^{\mathrm{i}\theta(p),\, z(p)}))$ where $p = (\tilde{z},\,\tilde{\theta}) \in [-1,1]\times[0,2\pi)$ and

$$\begin{cases} r(p) = \sqrt{2(1-\tilde{z})} \\ t(p) = \tilde{\theta} + \epsilon\frac{\pi}{2} \\ \rho(p) = \sqrt{1-\tilde{z}^2} \\ \theta(p) = \tilde{\theta} \\ z(p) = \tilde{z}. \end{cases}$$

Proposition 2.7. *The map* S_ϵ*, where* $\epsilon = \pm 1$*, is continuous and* S_ϵ *restricted to* $(-1,\,1)\times\mathbb{R}/2\pi\mathbb{Z}$ *is a diffeomorphism onto its image. If we let* $\Lambda_0^\epsilon := S_\epsilon([-1,\,1]\times\mathbb{R}/2\pi\mathbb{Z})$*, then* $\Lambda_0^1\cup\Lambda_0^2 = \Lambda_0$ *and*

$$\Lambda_0^1\cap\Lambda_0^2 = \Big(\{(0,\,0)\}\times\{(1,\,0,\,0)\}\Big)\cup\Big(C_2\times\{(0,\,0,\,-1)\}\Big),$$

where C_2 *denotes the circle of radius* 2 *centered at* $(0,\,0)$ *in* $\mathbb{R}^2$*. Moreover,* S_ϵ *restricted to* $(-1,\,1)\times\mathbb{R}/2\pi\mathbb{Z}$ *is a smooth Lagrangian embedding into* $\mathbb{R}^2\times S^2$*.*

Proof. On the one hand we have that $z^2 = 1-x^2-y^2 = 1-\rho^2$. The expressions for the maps J and H in the new coordinates $(r,\,t,\,\rho,\,\theta)$ are

$$J = \frac{1}{2}r^2 \pm \sqrt{1-\rho^2}, \qquad H = \frac{r\rho}{2}\cos(t-\theta). \tag{2.6}$$

In virtue of the formula for H in the right hand-side of (2.6), if $H = 0$ then $r = 0$ or $\rho = 0$ or $t-\theta = \frac{\pi}{2}(\mathrm{mod}\,\pi)$, which leads to three separate cases. The first case is when $r = 0$; then $J = \pm\sqrt{1-\rho^2} = 1$, and hence $\rho = 0$. Hence the only solution is $(u, v, x, y, z) = (0, 0, 0, 0, 1)$. The second case is when $\rho = 0$; then either $z = 1$ and $r = 0$, or $z = -1$ and $r = 2$. Hence the set of solutions consists of $(0, 0, 0, 0, 1)$ and the circle $r = 2$, $\rho = 0$ and $z = -1$. Finally, the third case is when $t - \theta = \frac{\pi}{2} (\operatorname{mod} \pi)$; because $J = 1$ and $H = 0$, it follows from the formula for z above and the left hand-side of (2.6) that $r^2 = 2(1 - z)$. Hence the set of solutions Λ_0 is equal to the set of points $(r\,\mathrm{e}^{\mathrm{i}t}, \rho\,\mathrm{e}^{\mathrm{i}\theta})$ such that

$$\begin{cases} r = \sqrt{2(1-z)}, \quad z \in [-1, 1] \\ \theta = t - \frac{\pi}{2} \quad \text{or} \quad \theta = t + \frac{\pi}{2}, \quad t \in [0, 2\pi) \\ \rho = \sqrt{1 - z^2} \end{cases} \tag{2.7}$$

This case contains the previous two cases, which proves statement (3) part (i) in virtue of expression (2.5). The other statements are left to the reader. □

Remark 2.8 The singular fiber Λ_0 consists of two sheets glued along a point and a circle; topologically Λ_0 is a pinched torus, i.e. a 2-dimensional torus $S^1 \times S^1$ in which one circle $\{p\} \times S^1$ is contracted to a point (which is of course not a a smooth manifold at the point which comes from the contracting circle). ⊘

The radial vector field $\mathcal{X}_{\mathcal{H}}$ on Λ_0.

Proposition 2.9. *Let $\mathcal{X}_{q_i}$ be the Hamiltonian vector field of q_i (which recall is defined in saturated neighborhood of the singular fiber Λ_0). On the singular fiber Λ_0, the vector fields $\mathcal{X}_{q_1}$, $\mathcal{X}_J$ and $\mathcal{X}_{q_2}$, $\mathcal{X}_H$ are linearly independent, precisely: $\mathcal{X}_{q_1} = \mathcal{X}_J, \quad \mathcal{X}_{q_2} = 2\,\mathcal{X}_H$. In particular the vector field $\mathcal{X}_H$ is radial.*

Proof. It follows from Eliasson's theorem that there exists a smooth function h such that $q = h \circ F$ and $\mathrm{d}h(0)$ is the invertible 2 by 2 matrix B in Lemma 2.6.

Then on Λ_0 we have that

$$\mathcal{X}_{q_i} = \frac{\partial h_i}{\partial J}\,\mathcal{X}_J + \frac{\partial h_i}{\partial H}\,\mathcal{X}_H, \quad i = 1, 2. \tag{2.8}$$

Because the coefficients are constant along Λ_0, it is sufficient to do the computation at the origin. At the origin the computation is given by the matrix B in Lemma 2.6, so we have that $\frac{\partial h_1}{\partial J}(0) =, \frac{\partial h_1}{\partial H}(0) = 0$, $\frac{\partial h_2}{\partial J}(0) = 0$ and $\frac{\partial h_2}{\partial H}(0) = 2$. The proposition follows from (2.8). □

In the following section we will need to use explicitly the Hamiltonian vector field $\mathcal{X}_H$, and therein it will be most useful to a have the following explicit coordinate expression.

Lemma 2.10. *The Hamiltonian vector field $\mathcal{X}_H$ of H is of the form*

$$\mathcal{X}_H = \frac{y}{2}\frac{\partial}{\partial u} - \frac{x}{2}\frac{\partial}{\partial v} + \frac{-yu + xv}{2}\frac{\partial}{\partial z} - \frac{z(xu + yv)}{2(1 - z^2)}\frac{\partial}{\partial \theta}.$$

Proof. For this computation let us use coordinates (u, v, z, θ) as a parametrization of $\mathbb{R}^2 \times S^2$.

The coordinate expression for the Hamiltonian H is $H = \frac{1}{2}(xu + yv) = \frac{1}{2}(\rho\cos\theta u + \rho\sin\theta v)$, Then the Hamiltonian vector field $\mathcal{X}_H$ is of the form $\mathcal{X}_H = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial z} + d\frac{\partial}{\partial \theta}$, where since the symplectic form on $\mathbb{R}^2 \times S^2$ in these coordinates is $\mathrm{d}u \wedge \mathrm{d}v + \mathrm{d}\theta \wedge \mathrm{d}z$, the function coefficient a (which will be important later in the proof) is given by

$$a = \frac{\partial H}{\partial v} = \frac{1}{2}\rho\sin(\theta) = \frac{y}{2} \tag{2.9}$$

and the other function coefficients are given by $b = -\frac{\partial H}{\partial u} = \rho\cos(\theta) = -\frac{x}{2}, c = \frac{\partial H}{\partial \theta} = \frac{\rho}{2}(-\sin(\theta)u + \cos(\theta)v) = \frac{-yu+xv}{2}$ and $d = -\frac{\partial H}{\partial z}$.

We need to compute d explicitly. Since $\frac{\partial \theta}{\partial z} = 0$ because the angle θ does not depend on the height z, and $\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\frac{z}{\sqrt{1-z^2}}$, we have that

$$\frac{\partial x}{\partial z} = \frac{\partial x}{\partial \rho}\frac{\partial \rho}{\partial z} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial z} = \frac{\partial x}{\partial \rho}\frac{\partial \rho}{\partial z} = \frac{-xz}{\rho^2} \tag{2.10}$$

$$\frac{\partial y}{\partial z} = \frac{\partial y}{\partial \rho}\frac{\partial \rho}{\partial z} + \frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial z} = \frac{\partial y}{\partial \rho}\frac{\partial \rho}{\partial z} = \frac{-yz}{\rho^2} \tag{2.11}$$

It follows that from (2.10) and (2.11) that the function coefficient d is given by

$$d = -\frac{\partial H}{\partial z} = -\frac{\partial H}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial H}{\partial y}\frac{\partial y}{\partial z} = \frac{u}{2}\frac{-xz}{\rho^2} + \frac{v}{2}\frac{-yz}{2\rho^2} = -\frac{z(xu+yv)}{2\rho^2} = -\frac{z(xu+yv)}{2(1-z^2)}.$$

□

Definition of a simple "radial" loop in Λ_0. In order to apply the theorem it is enough to take γ_0 to be an integral curve of the radial vector field $\mathcal{X}_H$.

We define γ_0 as the simple loop obtained through the parametrizations S_+ and S_- by letting $\tilde{z}$ run from -1 to 1 and back to -1, respectively. For instance, one can use the formula

$$\gamma_0(s) \quad := \quad \begin{cases} S_1(-1+4s, -\frac{\pi}{2}) & \text{if } 0 \leq s \leq \frac{1}{2}; \\ S_2(3-4s, \frac{\pi}{2}) & \text{if } \frac{1}{2} < s \leq 1. \end{cases}$$

Corollary 2.11. *Along the curve γ_0 we have*

$$\mathcal{X}_H\Big|_{\gamma_0} = \frac{y}{2}\frac{\partial}{\partial u} - \frac{yu}{2}\frac{\partial}{\partial z}. \tag{2.12}$$

Proof. We use the notation of Lemma 2.10. Along γ_0 we have $v = 0$, $x = 0$ and $\theta = \pi$ or $\theta = \frac{3\pi}{2}$. Hence $a = \frac{y}{2}, b = 0, , c = -\frac{yu}{2}, d = 0$. Therefore the vector field $\mathcal{X}_H$ along the curve γ_0 is given by (2.12). □

Using Corollary 2.11 we describe the very explicit relation between the curve γ_0 and the Hamiltonian vector field $\mathcal{X}_H$.

Proposition 2.12. *The curve $\gamma_0\colon [0,\, 1] \to M$ is an integral curve of $\mathcal{X}_H$.*

Proof. Since by construction the vector field $S_*(\frac{\partial}{\partial \tilde{z}})$ is tangent to the curve γ_0, it is enough to show that $S_*(\frac{\partial}{\partial \tilde{z}})$ is colinear to $\mathcal{X}_H$ are colinear at each point.

A computation gives that

$$S_*\Big(\frac{\partial}{\partial \tilde{z}}\Big) = \frac{\partial}{\partial z} - \frac{1}{\sqrt{2(1-z)}}\frac{\partial}{\partial r} + \frac{z}{\sqrt{1-z^2}}\frac{\partial}{\partial \rho}. \tag{2.13}$$

On the other hand

$$u = \sqrt{2(1-z)}, \tag{2.14}$$

and since $(r,\,t)$ are polar coordinates for $(u,\,v)$, $\frac{\partial}{\partial r} = \cos t\,\frac{\partial}{\partial u} + \sin t\,\frac{\partial}{\partial v}$, which at $t = 0$ gives that $\frac{\partial}{\partial r} = \frac{\partial}{\partial u}$. Therefore, because at $t = 0$ the last factor of (2.13) is zero, we conclude from (2.14) that

$$S_*\Big(\frac{\partial}{\partial \tilde{z}}\Big) = \frac{\partial}{\partial z} - \frac{1}{u}\,\frac{\partial}{\partial u}. \tag{2.15}$$

.

It follows from (2.12) that $\mathcal{X}_H = -\frac{yu}{2}\,S_*\Big(\frac{\partial}{\partial \tilde{z}}\Big)$, which shows that $\mathcal{X}_H$ and $S_*(\frac{\partial}{\partial \tilde{z}})$ are colinear at every point, as desired.

□

Stage 3 — Integration in linearized Eliasson's coordinates

Let ϕ be a local symplectic map such that $g \circ F \circ \phi = q$ on $\mathbb{R}^4$, as given by Eliasson's normal form theorem. The integrals in (..) are defined in terms of the corresponding canonical coordinates $(x_1,\,x_2,\,\xi_1,\,\xi_2)$ in $\mathbb{R}^4$.

Because our computation is local, we can use instead the linearized coordinates that we have defined in Lemma 2.6. More precisely, one can always choose ϕ such that the tangent map $\mathrm{d}_{(0,\,0,\,0,\,0)}\,\phi$: $\mathrm{T}_{(0,\,0,\,0,\,0)}\,\mathbb{R}^4 \to \mathrm{T}_{(0,\,0,\,1,\,0,\,0)}\,S^2 \times \mathbb{R}^2$ is equal to $\hat{\phi}$, and this gives local coordinates $(\hat{x}_1, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2)$ in a neighborhood of m, such that $B \circ F(\hat{x}_1, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2) = q(\hat{x}_1, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2) + \mathcal{O}(3)$.

Note that these coordinates are not symplectic, except at m.

Lemma 2.13. *The integral (2.5) gives us the same result when computed in linearized coordinates, i.e. upon replacing r_A by $\hat{r}_A$, t_A by $\hat{t}_A$, ρ_A by $\hat{\rho}_A$ and θ_A by $\hat{\theta}_A$.*

Proof. Since $r_A^2 = x_1^2 + x_2^2$, then

$$\hat{r}_A^2 = \hat{x}_1^2 + \hat{x}_2^2 = x_1^2 + x_2^2 + \mathcal{O}(3) = r_A^2 + \mathcal{O}(3) \tag{2.16}$$

We know that $\frac{\mathcal{O}(3)}{x_1^2+x_2^2} = \mathcal{O}(1)$, and therefore it follows from (2.16) that

$$\ln(\hat{r}_A^2) = \ln(r_A^2 + \mathcal{O}(3)) = \ln\Big(1 + \frac{\mathcal{O}(3)}{r_A^2}\Big) + \ln(r_A^2) = \ln(1 + \mathcal{O}(1)) + \ln(r_A^2) = \mathcal{O}(1) + \ln(r_A^2). \tag{2.17}$$

Similarly $\ln(\hat{\rho}_B^2) = \mathcal{O}(1) + \ln(\rho_B^2)$. Hence $\ln(r_A\rho_B) = \ln(r_A) + \ln(\rho_B) = \ln(\hat{r}_A) + \ln(\hat{\rho}_B) = \ln(\hat{r}_A\hat{\rho}_B) + \mathcal{O}(1)$. Then

$$\lim_{(s,\,t)\to(0,\,0)} \ln(r_{A_0}\rho_{B_0}) - \ln(\hat{r}_{A_0}\hat{\rho}_{B_0}) = 0. \tag{2.18}$$

It follows from expressions (2.5) and (2.18) that

$$a_2 = \lim_{(s_A,\,s_B)\to(0,\,0)} \Big(\int_{A_0=\gamma_0(s_A)}^{B_0=\gamma_0(1-s_B)} \kappa_{2,0} + \ln|\hat{r}_{A_0}\hat{\rho}_{B_0}|\Big). \tag{2.19}$$

This concludes the proof. □

Stage 4 — Computation of the first order Taylor series invariants a_1 and a_2

In order to compute the integrals in (2.19) we can replace γ_0 by any integral curve of $\mathcal{X}_H$ with the same endpoints. Thus, let γ be a solution to $\dot{\gamma} = \mathcal{X}_H \circ \gamma$. By definition, for any 1-form κ,

$$\int_{A_0:=\gamma(s_1),\ \text{along}\,\gamma}^{B_0:=\gamma(s_2)} \kappa = \int_{s_1}^{s_2} \kappa_{\gamma(s)}(\dot{\gamma}(s))\,\mathrm{d}s = \int_{s_1}^{s_2} \kappa_{\gamma(s)}(\mathcal{X}_H(\gamma(s)))\,\mathrm{d}s. \tag{2.20}$$

Theorem 2.14. *Let $(S) \in \mathbb{R}[[X, Y]]$ be the Taylor series invariant of the couple-spin oscillator. Then the first coefficient of the first term of the series is given by $a_1 = \frac{\pi}{2}$. The second coefficient of the first term of the first order Taylor series invariant is $a_2 = 5\ln 2$.*

Proof. We have divided the computation of a_2 in several steps.

Step 1: Set-up of the integral of $\kappa_{2,0}$. We need to compute expression (2.19).

Let a be given by (2.9).

In view of (2.12), the path γ between A_0 and B_0 can be parametrized by the variable u. This means that the path γ is obtained by first increasing u up to $u = 2$ on the first sheet (parametrized by S_1) and then decreasing u on the second sheet (parametrized by S_2).

By Lemma 2.9 we know that $\mathcal{X}_{q_2} = 2\mathcal{X}_H$ and hence $(\kappa_{2,0})_{\gamma(s)}(\mathcal{X}_H(\gamma(s))) = \frac{(\kappa_{2,0})_{\gamma(t)}(\mathcal{X}_{q_2}(\gamma(s)))}{2}$. By definition of $\kappa_{2,0}$ we know that $\kappa_{2,0}(\mathcal{X}_{q_2}) = -1$ and hence it follows from (2.20) that $\int_{A_0,\,\text{along}\,\gamma}^{B_0} \kappa_{2,0} = \int_{s_1}^{s_2} \frac{\mathrm{d}s}{2}$. Since $\frac{\mathrm{d}u}{\mathrm{d}s}$ is equal to $a = \frac{y}{2}$ we have that

$$\int_{A_0,\ \text{along}\,\gamma}^{B_0} \kappa_{2,0} = \int_{s_1}^{s_2} \frac{\mathrm{d}s}{2} = \int_{u_1}^{2} \frac{\mathrm{d}u}{y_+(u)} + \int_{2}^{u_2} \frac{\mathrm{d}u}{y_-(u)}, \tag{2.21}$$

where $y_\pm(u)$ is the y-coordinate along the part of the curve γ_0 which corresponds to the parametrization $S_\pm$, respectively. Our next goal is to compute expression (2.21).

Step 2: Computation of expression (2.21). Now, $y = \rho\sin(\theta) = \pm\rho$.

Now let us express the dependence of y in u along the path γ. By the equation $J = \frac{1}{2}(u^2+v^2)+z = -1$, which is always true along the singular fiber, we have that, since $v = 0$, $\frac{u^2}{2} + z = 1$, or in other words, $z = 1 - \frac{u^2}{2}$. It follows from this equation that

$$y_\pm = \pm\rho = \pm\sqrt{1-z^2} = \pm\sqrt{1-(1-\frac{u^2}{2})^2} = \pm u\sqrt{1-\frac{u^2}{4}} \quad \text{since } u > 0. \tag{2.22}$$

On the other hand, note that the function $G(t) = \ln\left(\frac{1}{\cos t} + \tan t\right)$ is a primitive of the function $g(t) = \frac{1}{\cos t}$. Then by equation (2.22), using the change of variable $u/2 = \cos t$, and then applying the fundamental theorem of calculus we obtain[3]

$$\int_{u_1}^{2} \frac{\mathrm{d}u}{y_+} = \int_{u_1}^{2} \frac{\mathrm{d}u}{u\sqrt{1-\frac{u^2}{4}}} = -\Big[\ln\Big(\frac{1}{\cos t} + \tan t\Big)\Big]_{t_1}^{0} = -\Big[\ln\Big(\frac{2}{u} + \frac{2}{u}\sqrt{1-\frac{u^2}{4}}\Big)\Big]_{u_1}^{2},$$

[3] The integral is equal to 0 when $u = 2$

and simplifying this expression we then obtain

$$\int_{u_1}^{2} \frac{\mathrm{d}u}{y_+} = \ln\Big(\frac{2}{u_1}\Big) + \ln\Big(1+\sqrt{1-\frac{u_1^2}{4}}\Big). \tag{2.23}$$

The goal of this proof is to compute a_1, which by (2.5) is equal to the limit

$$\lim_{(s,\,t)\to(0,\,0)}\Big(\int_{A_0:=\gamma_0(s)}^{B_0(t):=\gamma_0(1-t)} \kappa_{2,0} + \ln(r_{A_0}\rho_{B_0})\Big),$$

and precisely because this limit exists, we may calculate it along the diagonal values given by $u = u_1 = u_2$. Then it follows from equation (2.23) that

$$\int_{A_0}^{B_0} \kappa \;=\; \int_{u_1}^{2} \frac{\mathrm{d}u}{y_+} + \int_{2}^{u_2} \frac{\mathrm{d}u}{y_-} = 2\int_{u}^{2} \frac{\mathrm{d}y_+}{y_+} = 2\Big(\ln\Big(\frac{2}{u}\Big) + \ln\Big(1+\sqrt{1-\frac{u^2}{4}}\Big)\Big). \tag{2.24}$$

This concludes this step.

<u>Step 3</u>: *Computation of the logarithm factor* $\ln(\hat r_{A_0}\hat\rho_{B_0})$.

From the notation of Stage 1 we have that $\hat r_A^2 = \hat x_1^2 + \hat x_2^2$ and that $\hat\rho_A^2 = \hat\xi_1^2 + \hat\xi_2^2$. Using Lemma 2.6 we find that $\hat r_A^2 = \frac{1}{2}(x^2+y^2+u^2+v^2) + (-uy+vx)$ and $\hat\rho_A^2 = \frac{1}{2}(x^2+y^2+u^2+v^2) + (uy-vx)$.

We need to compute $\hat r_{A_0}$ and $\hat\rho_{B_0}$. The points A_0 and B_0 are in the path γ_0 and $A_0 := (u_{A_0},\, v_{A_0},\, \theta_{A_0},\, z_{A_0}) = (u_{A_0},\, 0,\, \frac{\pi}{2},\, 1-\frac{u_{A_0}^2}{2})$, and $B_0 := (u_{B_0},\, v_{B_0},\, \theta_{B_0},\, z_{B_0}) = (u_{A_0},\, 0,\, \frac{-\pi}{2},\, 1-\frac{u_{A_0}^2}{2})$.

With this information we can compute $\hat r_{A_0}$ and $\hat\rho_{B_0}$ using expression (2.22) and recalling that $x = v = 0$ along γ:

$$\hat r_{A_0}^2 = \frac{1}{2}(u^2 - \frac{u^4}{4} + u^2) - u^2\sqrt{1-\frac{u^2}{4}} \;=\; \frac{u^2}{2}(2 - \frac{u^2}{4} - 2\sqrt{1-\frac{u^2}{4}}), \tag{2.25}$$

where here we have also used $\rho^2 = 1 - z^2 = 1 - (1-\frac{u^2}{2})^2 = u^2 - \frac{u^2}{4}$. And we also have that

$$\hat\rho_{B_0}^2 = \hat r_{A_0}^2. \tag{2.26}$$

It follows from (2.25) and (2.26) that

$$\ln(\hat r_{A_0}\hat\rho_{B_0}) = \frac{1}{2}\ln(\hat r_{A_0}^2\hat\rho_{B_0}^2) = \frac{1}{2}\ln(\hat r_{A_0}^4) = \ln(\hat r_{A_0}^2) = \ln\Big(\frac{u^2}{2}(2 - \frac{u^2}{4} + 2\sqrt{1-\frac{u^2}{4}})\Big)$$

and therefore that

$$\ln(\hat r_{A_0}\hat\rho_{B_0}) = 2\ln(\frac{u}{\sqrt{2}}) + \ln(2 - \frac{u^2}{4} + 2\sqrt{1-\frac{u^2}{4}}). \tag{2.27}$$

This concludes the computation of the logarithmic factor.

<u>Step 4</u>: *Conclusion.* It follows from (2.5), (2.24) and (2.27) that

$$\begin{aligned} a_2 &= \lim_{u\to 0}\Big(\int_{A_0}^{B_0} \kappa_{2,0} + \ln(\hat r_{A_0}\hat\rho_{B_0})\Big) \\ &= \lim_{u\to 0}\Big((2\ln(\frac{2}{u}) + 2\ln(1+\sqrt{1-\frac{u^2}{4}}) + 2\ln(\frac{u}{\sqrt{2}}) + \ln(2 - \frac{u^2}{4} + 2\sqrt{1-\frac{u^2}{4}})\Big) \\ &= 2\ln 2 + 2\ln 2 - \ln 2 + 2\ln 2 = 5\ln 2. \end{aligned} \tag{2.28}$$

So we have proven that $a_2 = 5\ln 2$ as we wanted to show.

In order to find a_1, note that the following hold: $u \geq 0,\ v = 0,\ \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2},\ \rho = \sqrt{1-z^2},\ z = 1 - \frac{u^2}{4},\ \rho = \sqrt{u^2 - \frac{u^2}{4}}$. In this case $x_1 = \frac{u \pm \rho}{2},\quad x_2 = \frac{u \pm \rho}{2}$, and therefore $\hat{\theta} = \frac{\pi}{4}$. Similarly $\xi_1 = \frac{-u \pm \rho}{2},\quad \xi_2 = \frac{u \mp \rho}{2} = -\xi_1$, and hence $\alpha = \frac{\pi}{4}$. It follows that $\hat{\theta}_{A_0} - \hat{\alpha}_{B_0} = \frac{\pi}{2}$. Therefore by Theorem 2.5

$$a_1 = \lim_{(s,\,t) \to (0,\,0)} \Big(\int_{A_0 = \gamma_0(1)}^{B_0 = \gamma_0(1-t)} \kappa_{1,0} + (\hat{\theta}_A - \hat{\alpha}_B) \Big) = \frac{\pi}{2}.$$

Here we are using that because $\kappa_0(\mathcal{X}_H) = 0$ and γ_0 is tangent everywhere to $\mathcal{X}_H$ so one has that

$$\lim_{(s,\,t) \to (0,\,0)} \Big(\int_{A_0 = \gamma_0(1)}^{B_0 = \gamma_0(1-t)} \kappa_0 \Big) = 0.$$

(See also the paragraphs before Theorem 2.14). This concludes the proof. $\square$

Theorem 1.1 follows from Theorem 2.14.

Remark 2.15 It is plausible that our proof technique generalizes to compute the higher order terms of the Taylor series invariant, but not immediately, as we rely on the limit theorem proved in [22] which only applies to the first two terms. The computation provides more evidence of the fact that from a dynamical and geometric view-point focus-focus singularities contain a large amount of information. $\oslash$

3 Convexity theory for coupled spin-oscillators

The plane $\mathbb{R}^2$ is equipped with its standard affine structure with origin at $(0,0)$, and orientation. Let $\mathrm{Aff}(2,\mathbb{R}^2) := \mathrm{GL}(2,\mathbb{R}^2) \ltimes \mathbb{R}^2$ be the group of affine transformations of $\mathbb{R}^2$. Let $\mathrm{Aff}(2,\mathbb{Z}) := \mathrm{GL}(2,\mathbb{Z}) \ltimes \mathbb{R}^2$ be the subgroup of *integral-affine* transformations.

Let $\mathcal{T}$ be the subgroup of $\mathrm{Aff}(2,\,\mathbb{Z})$ of those transformations which leave a vertical line invariant, or equivalently, an element of $\mathcal{T}$ is a vertical translation composed with a matrix T^k, where $k \in \mathbb{Z}$ and

$$T^k := \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in \mathrm{GL}(2,\,\mathbb{Z}).$$

Let $\ell_0 \subset \mathbb{R}^2$ be a vertical line in the plane, not necessarily through the origin, which splits it into two half--spaces, and let $n \in \mathbb{Z}$. Fix an origin in ℓ. Let $t^n_{\ell_0} : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity on the left half-space, and T^n on the right half-space. By definition $t^n_{\ell_0}$ is piecewise affine. A *convex polygonal set* Δ is the intersection in $\mathbb{R}^2$ of (finitely or infinitely many) closed half-planes such that on each compact subset of the intersection there is at most a finite number of corner points. We say that Δ is *rational* if each edge is directed along a vector with rational coefficients. For brevity, in this paper we usually write "*polygon*" instead of "*convex polygonal set*".

3.1 Construction of the semitoric polygon invariant

Let ℓ be a vertical line through the focus-focus value c. Let $B_{\mathrm{r}} := \mathrm{Int}(B) \setminus \{c\}$, which is precisely the set of regular values of F. Given a sign $\epsilon \in \{-1,+1\}$, let $\ell^\epsilon \subset \ell$ be the vertical half line starting at c at extending in the direction of ϵ : upwards if $\epsilon = 1$, downwards if $\epsilon = -1$.

In Th. 3.8 in [25] it was shown that for $\epsilon \in \{-1,+1\}$ there exists a homeomorphism $f = f_\epsilon \colon B \to \mathbb{R}^2$, modulo a left composition by a transformation in $\mathcal{T}$, such that $f|_{(B \setminus \ell^\epsilon)}$ is a diffeomorphism into its image $\Delta := f(B)$, which is a *rational convex polygon*, $f|_{(B_r \setminus \ell^\epsilon)}$ is affine (it sends the integral affine structure of B_{r} to the standard structure of $\mathbb{R}^2$) and f preserves J: i.e. $f(x,\, y) = (x,\, f^{(2)}(x,\, y))$. f satisfies further properties [18], which are relevant for the uniqueness theorem proof. In order to arrive at Δ one cuts $(J,\, H)(M) \subset \mathbb{R}^2$ along the vertical half-lines ℓ^ϵ. Then the resulting image becomes simply connected and thus there exists a global 2-torus action on the preimage of this set. The polygon Δ is just the closure of the image of a toric momentum map corresponding to this torus action.

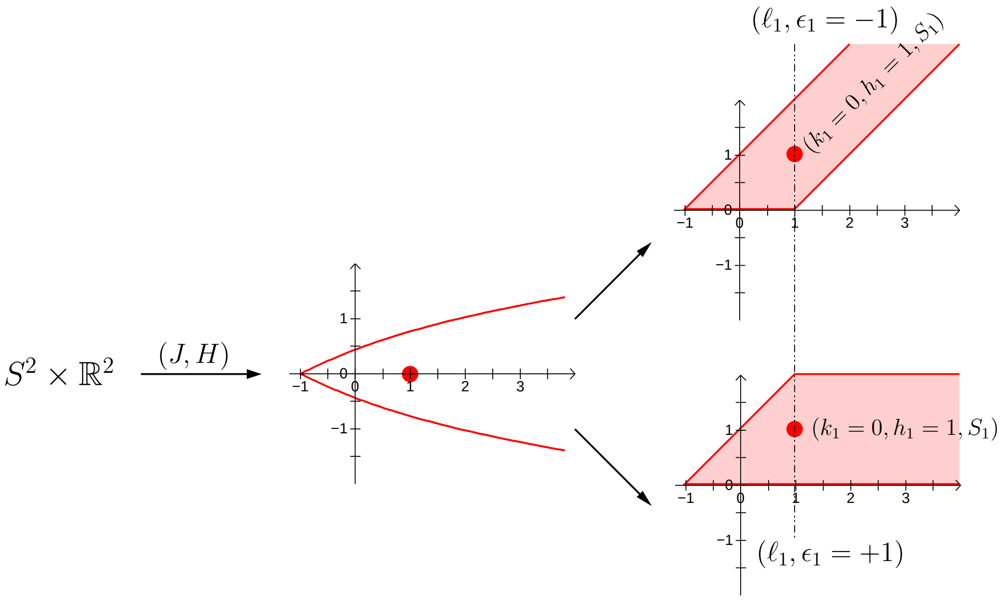

Figure 3.1: The coupled spin-oscillator example. The middle figure shows the image of the initial moment map $F = (J,\, H)$. Its boundary is the parametrized curve $(j(s) = \frac{s^2-3}{2s}, h(s) = \pm\frac{s^2-1}{2s^{3/2}}),\ s \in [1,\infty)$. The image is the connected component of the origin. The system is a simple semitoric system with one focus-focus point whose image is $(1,\, 0)$. The invariants are depicted on the right hand-side. The class of generalized polygons for this system consists of two polygons.

We can see that this polygon is not unique. The choice of the "cut direction" is encoded in the signs ϵ, and there remains some freedom for choosing the toric momentum map. Precisely, the choices and the corresponding homeomorphisms f are the following :

(a) *an initial set of action variables* f_0 *of the form* $(J,\, K)$ near a regular Liouville torus in [25, Step 2, pf. of Th. 3.8]. If we choose f_1 instead of f_0, we get a polygon Δ' obtained by left composition with an element of $\mathcal{T}$. Similarly, if we choose f_1 instead of f_0, we obtain f composed on the left with an element of $\mathcal{T}$;

(b) *an integer* $\epsilon \in \{1,\, -1\}$. If we choose ϵ' instead of ϵ we get $\Delta' = t_u(\Delta)$ with $u = (\epsilon - \epsilon')/2$, by [25, Prop. 4.1, expr. (11)]. Similarly instead of f we obtain $f' = t_u \circ f$.

Once f_0 and ϵ have been fixed as in (a) and (b), respectively, then there exists a unique toric momentum map μ on $M_r := F^{-1}(\mathrm{Int}B \setminus \ell\epsilon)$ which preserves the foliation $\mathcal{F}$, and coincides with $f_0 \circ F$ where they are both defined. Then, necessarily, the first component of μ is J, and we have $\overline{\mu(M_r)} = \Delta$.

We need now for our purposes to formalize choices (a) and (b) in a single geometric object. Let $\mathrm{Polyg}(\mathbb{R}^2)$ be the space of rational convex polygons in $\mathbb{R}^2$. Let $\mathrm{Vert}(\mathbb{R}^2)$ be the set of vertical lines in $\mathbb{R}^2$. A *weighted polygon* (of complexity 1) is a triple of the form $\Delta_{\mathrm{w}} = \Big(\Delta,\, \ell_\lambda,\, \epsilon\Big)$ where $\Delta \in \mathrm{Polyg}(\mathbb{R}^2)$, $\ell \in \mathrm{Vert}(\mathbb{R}^2)$, and $\epsilon \in \{-1,\, 1\}$. Let $G := \{-1,\, +1\}$. Obviously, the group $\mathcal{T}$ sends a rational convex polygon to a rational convex polygon. It corresponds to the transformation described in (a). On the other hand, the transformation described in (b) can be encoded by the group G acting on the triple Δ_{w} by the formula

$$\epsilon' \cdot \Big(\Delta,\, \ell_\lambda,\, \epsilon\Big) = \Big(t_u(\Delta),\, \ell_\lambda,\, \epsilon'\,\epsilon\Big),$$

where $\vec{u} = (\epsilon - \epsilon')/2$. This, however, does not always preserve the convexity of Δ, as is easily seen when Δ is the unit square centered at the origin and $\lambda_1 = 0$. However, when Δ comes from the construction described above for a semitoric system $(J,\, H)$, the convexity is preserved. Thus, we say that a weighted polygon is *admissible* when the G-action preserves convexity. We denote by $\mathcal{W}\mathrm{Polyg}(\mathbb{R}^2)$ the space of all admissible weighted polygons (of complexity 1). The set $G \times \mathcal{T}$ is an abelian group, with the natural product action. The action of $G \times \mathcal{T}$ on $\mathcal{W}\mathrm{Polyg}(\mathbb{R}^2)$, is given by:

$$(\epsilon',\, \tau) \cdot \Big(\Delta,\, \ell_\lambda,\, \epsilon\Big) = \Big(t_u(\tau(\Delta)),\, \ell_\lambda,\, \epsilon'\,\epsilon\Big),$$

where $u = (\epsilon - \epsilon')/2$. We call a *semitoric polygon* the equivalence class of an admissible weighted polygon under the $(G \times \mathcal{T})$-action.

Let Δ be a rational convex polygon obtained from the momentum image $(J,\, H)(M)$ according to the above construction of cutting along the vertical half-line ℓ^ϵ.

Definition 3.1 The *semitoric polygon invariant of* $(M,\, \omega,\, (J,\, H))$ is the semitoric polygon equal to the $(G \times \mathcal{T})$-orbit $(G \times \mathcal{T}) \cdot \Big(\Delta,\, \ell,\, \epsilon\Big) \in \mathcal{W}\mathrm{Polyg}(\mathbb{R}^2)/(G \times \mathcal{T})$. ⊘

3.2 The semitoric polygon invariant of coupled spin-oscillators

Proposition 3.2. *The semitoric polygon invariant of the coupled spin-oscillator is the $(G \times \mathcal{T})$-orbit consisting of the two convex polygons depicted on the right hand-side of Figure 3.1.*

Proof. As shown in Figure 3.1, a representative of the semitoric polygon invariant is a polygon in $\mathbb{R}^2$ with exactly two vertices at $(-1,\, 0)$ and $(1,\, 0)$, and from these two points leave straight lines with slope 1 (the other possible polygon representative has vertices at $(-1,\, 0)$ and $(1,\, 2)$). One finds this polygon simply by combining the information about the isotropy weights at the left corner of the polygon (an elliptic-elliptic critical value) [25, Prop. 6.1], together with the formula given in [25, Thm. 5.3], in which the relation between isotropy weights and the slopes of the edges of the polygon is described using the Duistermaat-Heckman function. □

3.3 Classification theory for coupled spin-oscillators

The authors have recently given a general classification of general semitoric integrable in dimesion 4 [18], [19] in terms of five symplectic invariants; the reader familiar with these works can easily that two of these invariants do not appear in the case of coupled spin-oscillators, and we state the uniqueness theorem therein in this particular case[4]

Consider a focus-focus critical point m whose image by $(J,\,H)$ is $\tilde{c}$, and let Δ be a rational convex polygon corresponding to the system $(M,\,\omega,\,(J,\,H))$. If μ is a toric momentum map for the system $(M,\,\omega,\,(J,\,H))$ corresponding to Δ, then the image $\mu(m)$ is a point in the interior of Δ, along the line ℓ. We proved in [18] that the vertical distance $h := \mu(m) - \min_{s\in\ell\cap\Delta}\pi_2(s) > 0$ is independent of the choice of momentum map μ. Here $\pi_2\colon \mathbb{R}^2 \to \mathbb{R}$ is $\pi_2(c_1,\,c_2) = c_2$.

Theorem 3.3 (consequence of Th. 6.2, [18])**.** *Let $(M,\,\omega,\,(J,\,H))$ be a 4-dimensional semitoric integrable system with exactly one focus-focus singularity. The* list of invariants of $(M,\,\omega,\,(J,\,H))$ *consists of the following items: (i) the Taylor series invariant $(S)^\infty$ at the focus-focus singularity m; (ii) the semitoric polygon invariant; (iii) the volume invariant, i.e. the height $h > 0$ of m. Two 4-dimensional simple semitoric integrable systems $(M_1,\,\omega_1,(J_1,\,H_1))$ and $(M_2,\,\omega_2,(J_2,\,H_2))$ with exactly one focus-focus singularity are isomorphic if and only if the list of invariants (i)-(iii) of $(M_1,\,\omega_1,(J_1,\,H_1))$ is equal to the list of invariants (i)-(iii) of $(M_2,\,\omega_2,(J_2,\,H_2))$.*

Theorem 3.4. *The coupled spin-oscillator has the following symplectic invariants: (i) first terms of the Taylor series invariant: $a_1 = \frac{\pi}{2}$ and $a_2 = 5\ln 2$; (ii) semitoric polygon invariant: $(G\times\mathcal{T})\cdot\Delta_{\mathrm{w}}$, where Δ_{w} is either the upper or lower weighted polygon depicted on the right-most side of Figure 3.1; (iii) volume invariant: $h = 1$.*

Proof. The semitoric polygon invariant and the first terms of the Taylor series invariant were computed previously. The height of the focus-focus point of the system in the polygon is equal to half of the Liouville volume of the submanifold of M given by the equation $J = 1$. This is because the functions H and J are symmetric about the J-axis of $\mathbb{R}^2$ in the sense that $J(x,\,y,\,z,\,u,\,v) = J(x,\,y,\,z,\,-u,\,-v)$ and $H(x,\,y,\,z,\,u,\,v) = -H(x,\,y,\,z,\,-u,\,-v)$. Here there is no need to compute anything because the volume of the submanifold given by $J = 1$ in M is just the length of the vertical slice of the polygon at $J = 1$, which is 2, and hence the height of the focus-focus point of the system is $h_1 = 1$, and the image of the focus-focus point in the polygon is $(1,\,1)$. ☐

4 Spectral theory for quantum spin-oscillators

In this section, we use the notation of the previous sections $J = \frac{u^2+v^2}{2} + z$ and $H = \frac{1}{2}(xu + vy)$. Our goal in this section is to quantize this example and analyze its semiclassical spectrum.

First we quickly review the process of assigning a quantum system to a classical system. Loosely speaking, a *quantum integrable system* is a collection of commuting self-adjoint operators on a Hilbert space. *Quantization* is a process that takes a classical phase space (here, a symplectic manifold M) to a Hilbert space $\hat{M}$, and classical Hamiltonians $f \in \mathrm{C}^\infty(M)$ to self-adjoint operators $\hat{f}$ acting on $\hat{M}$. The quantization of symplectic manifold is often called geometric quantization. See the recent book by Kostant-Pelayo [15] for a survey. Quantizing Hamiltonians involves more difficulties. For instance, we need the map

[4]The first of these invariants is the number of focus-focus singularities. The last of these invariants, the so called twisting index invariant, is a rather subtle topological invariant which measures how the topology near a focus-focus singular fiber relates to the topology near the other focus-focus fibers. Hence the invariant only appears when there is more than one focus-focus singularity, and in the following we shall not mention it. The twisting-index expresses the fact that there is, in a neighborhood of any focus-focus point c_i, a *privileged toric momentum map* ν. This momentum map, in turn, is due to the existence of a unique hyperbolic radial vector field in a neighborhood of the focus-focus fiber. Therefore, one can view the twisting-index as a dynamical invariant. This is an important invariant in the general case, see [18].

$f \mapsto \hat{f}$ to be a Lie algebra homomorphism, at least at first order : if the classical system is given by two Poisson commuting functions $f,\, g$ then the quantum system is given by two operators $\hat{f},\, \hat{g}$ such that

$$\tfrac{\hbar}{i}[\hat{f},\, \hat{g}] = 0 \qquad \mod\ (\mathcal{O}(\hbar)). \tag{4.1}$$

Such a quantization is well-known[5] to exist when $M = \mathbb{R}^{2n}$, and more generally on a cotangent bundle $M = \mathrm{T}^*X$, using $\hbar$-pseudodifferential quantization [7]. Quantizing compact symplectic manifolds is also possible under an integrality condition (the existence of a so-called prequantum line bundle), using Toeplitz quantization [4]. However, because of the remainder in (4.1), it is not known whether a classical integrable system can always be quantized to a true quantum integrable system. Very recently, in the algebraic setting, the relevant obstruction was defined [11]. In the coupled spin-oscillator example, like in many known systems, an exact quantization can be found by hand.

A well-known example is the harmonic oscillator in $\mathbb{R}^2$. The harmonic oscillator is given by $M = \mathbb{R}^2$ with coordinates $(u,\, v)$ and Hamiltonian function on it $N(u,\, v) = \frac{u^2+v^2}{2}$. The self-adjoint operator $\hat{N}$ in the Hilbert space $\mathrm{L}^2(\mathbb{R})$ given by $\hat{N} = -\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \frac{u^2}{2}$ is the standard Weyl quantization of the Hamiltonian N. The spectrum of $\hat{N}$ is discrete and given by $\{\hbar(n+\frac{1}{2}) \mid n \in \mathbb{N}\}$. The eigenfunctions are *Hermite functions*. This operator will be used as a quantum building tool in the sequel.

4.1 Quantization of $\mathbb{R}^4$ and the Harmonic Oscillator

We shall view S^2 as a reduced space of $\mathbb{R}^4 \simeq \mathbb{C}^2$ under the coordinate identification $z_1 = x_1 + \mathrm{i}\xi_1$, $z_2 = x_2 + \mathrm{i}\xi_2$. On $\mathbb{R}^4$ we consider the well-known harmonic oscillator, $L(z_1,\, z_2) = \frac{|z_1|^2+|z_2|^2}{2}$ which has a 2π-periodic flow generating a Hamiltonian S^1-action $t \cdot (z_1,\, z_2) = (z_1\,\mathrm{e}^{-\mathrm{i}t},\, z_2\,\mathrm{e}^{-\mathrm{i}t})$.

The space $Y_E := \{L = E\}$, for any value $E > 0$, is of course the euclidean 3-sphere $S^3_{\sqrt{2E}} \subset \mathbb{R}^4$ of radius $\sqrt{2E}$. It is well known that the reduced space $\{L = E\}/S^1$ is 2-sphere, and the fibration map $\{L = E\} \to \{L = E\}/S^1$ is the standard *Hopf fibration*. More precisely, we may represent this reduced space as the euclidean sphere $S^2_{E/2} \subset \mathbb{R}^3$ of radius $E/2$. Denoting by (x, y, z) the variables in $\mathbb{R}^3$, we have the following useful formula for the Hopf map, which will be used for quantization :

$$\begin{aligned} x &= \Re(z_1\bar{z}_2)/2 \\ y &= \Im(z_1\bar{z}_2)/2 \\ z &= (|z_1|^2 - |z_2|^2)/4. \end{aligned}$$

The usual quantization of $\mathbb{R}^4$ is the Hilbert space $\mathcal{H}_{\mathbb{R}^4} = \mathrm{L}^2(\mathbb{R}^2)$. The Weyl quantization of the Hamiltonian function L is the unbounded operator $\hat{L} := -\frac{\hbar^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} + \frac{\mathrm{d}^2}{\mathrm{d}x_2^2}\right) + \frac{x_1^2+x_2^2}{2}$.

The spectrum of $\hat{L}$ is given by $\mathrm{spec}(\hat{L}) = \{\hbar(n+1) \,|\, n \in \mathbb{N}\}$. To see this, define the operator $\hat{L}_j := -\frac{\hbar^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\right) + \frac{x_j^2}{2}$ acting on $\mathrm{L}^2(\mathbb{R}_{x_j})$. We can write $\hat{L} = \hat{L}_1 + \hat{L}_2$. Note that the spectrum of $\hat{L}_j$ is

$$\mathrm{spec}(L_j) = \{\hbar(n_j + \frac{1}{2}) \,|\, n_j \in \mathbb{N}\}. \tag{4.2}$$

Therefore we deduce that the spectrum of $\hat{L}$ is given by $\{\hbar(n_1+n_2+1) \,|\, n_1 \in \mathbb{N},\, n_2 \in \mathbb{N}\}$, and the formula above follows since n_1 and n_2 are arbitrary non-negative integers. The multiplicity of $\hbar(n+1)$ is given by the number of pairs $(n_1,\, n_2)$ such that $n_1 + n_2 = n$, which is precisely $n + 1$.

[5] for instance Weyl quantization, but there are other possible choices

4.2 Quantization of the space $S^2 \times \mathbb{R}^2$ and the Hamiltonians J and L

We define the *quantization* of $S^2_{E/2}$ to be the finite dimensional Hilbert space $\mathcal{H}_E := \ker(\hat{L} - E)$. When $E = \hbar(n+1)$, then $\dim(\mathcal{H}_E) = n+1$ (otherwise $\mathcal{H}_E = \{0\}$). It will be convenient to introduce the "anihilation operators" $a_i := \frac{1}{\sqrt{2\hbar}}\Big(\hbar\frac{\partial}{\partial x_j} + x_j\Big)$, $i = 1, 2$, which naturally quantize $z_i/\sqrt{2\hbar}$, $i = 1, 2$ respectively. Then $\hat{L} = \hbar(a_1 a_1^* + a_2 a_2^* - 1)$. The *quantization* of the Hamiltonians x, y, z on $S^2_{E/2}$ are the restrictions to $\mathcal{H}_E$ of the operators:

$$\hat{x} := \frac{\hbar}{2}(a_1 a_2^* + a_2 a_1^*), \qquad \hat{y} := \frac{\hbar}{2\mathrm{i}}(a_1 a_2^* - a_2 a_1^*), \qquad \hat{z} := \frac{\hbar}{2}(a_1 a_1^* - a_2 a_2^*). \tag{4.3}$$

This definition makes sense because $\mathcal{H}_E$ is stable under the action of $\hat{x}, \hat{y}, \hat{z}$. This can be checked right away using the commutation relations $[a_j, a_j^*] = 1$, but it will also follow from the explicit action of these operators, as explained in Section 4.3 below.

Of course, in $\mathbb{R}^2_{(u,v)}$, the quantization of v is $\hat{v} := (\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial u})$ and the quantization $\hat{u}$ of u is the multiplication by u (that we simply denote by u). Thus we have the very natural definition:

Definition 4.1 The *quantization* of $S^2_{E/2} \times \mathbb{R}^2$ is the (infinite dimensional) Hilbert space $\mathcal{H}_E \otimes \mathrm{L}^2(\mathbb{R}) \subset \mathrm{L}^2(\mathbb{R}^2) \otimes \mathrm{L}^2(\mathbb{R})$. The *quantization of* J is the operator $\hat{J} = \mathrm{Id} \otimes \Big(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial u^2} + \frac{u^2}{2}\Big) + (\hat{z} \otimes \mathrm{Id})$. The *quantization of* H is the operator $\hat{H} = \frac{1}{2}(\hat{x} \otimes u + \hat{y} \otimes (\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial u}))$. $\oslash$

This definition depends on the energy E, which will be fixed throughout the paper. For the numerical computations, we have taken $E = 2$, which corresponds to the quantization of the standard sphere $x^2 + y^2 + z^2 = 1$.

Lemma 4.2. *The operators $\hat{H}$ and $\hat{J}$ commute, i.e. we have the identity $[\hat{H},\, \hat{J}] = 0$, both in the functional analysis sense (*ie. *as an unbounded operator on a dense domain), and in the algebraic sense, as a bracket in the Lie algebra of polynomial differential operators.*

Proof. It is enough to show that $[\hat{H},\, \hat{J}] = 0$ holds on elements of the form $f \otimes g$, where f is any element in $\mathcal{H}_E$, and $g \in \mathrm{C}_0^\infty(\mathbb{R})$. And indeed,

$$\begin{aligned}
[\hat{H},\, \hat{J}](f \otimes g) &= (\hat{H}\hat{J} - \hat{J}\hat{H})(f \otimes g) = \hat{H}\hat{J}(f \otimes g) - \hat{J}\hat{H}(f \otimes g) \\
&= \hat{H}(f \otimes \hat{N}g + (\hat{z}f) \otimes g) - \frac{\hat{J}}{2}(\hat{x}f \otimes ug + \hat{y}f \otimes \hat{v}g) \\
&= \frac{1}{2}(\hat{x}f \otimes u\hat{N}g + \hat{x}\hat{z}f \otimes ug + \hat{y}f \otimes \hat{v}Ng + \hat{y}\hat{\xi}f \otimes \hat{v}g) \\
&- \frac{1}{2}(\hat{x}f \otimes \hat{N}ug + \hat{y}f \otimes \hat{N}\hat{v}g + \hat{z}\hat{x}f \otimes ug + \hat{z}\hat{y}f \otimes \hat{v}g) \\
&= \hat{x}f \otimes [u,\, \hat{N}]g + [\hat{x},\, \hat{z}]f \otimes ug + \hat{y}f \otimes [\hat{v},\, \hat{N}]g + [\hat{y},\, \hat{z}]f \otimes \hat{v}g.
\end{aligned} \tag{4.4}$$

As before, we have denoted $\hat{N} := -\frac{\hbar^2}{2}\frac{\partial^2}{\partial u^2} + \frac{u^2}{2}$. Now

$$[u,\, \hat{N}]f = u\Big(-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \frac{u^2}{2}\Big)f - \Big(-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \frac{u^2}{2}\Big)uf = \frac{\hbar^2}{2}\Big(-u\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \frac{\mathrm{d}^2}{\mathrm{d}\,u^2}u\Big)f$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}(uf) = f\frac{\mathrm{d}^2 u}{\mathrm{d}u^2} + 2\frac{\mathrm{d}\,f}{\mathrm{d}u}\frac{\mathrm{d}\,u}{\mathrm{d}u} + u\frac{\mathrm{d}^2 f}{\mathrm{d}u^2} = 2\frac{\mathrm{d}f}{\mathrm{d}u} + u\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}.$$

Hence $[u,\, \hat{N}]f = \frac{\hbar^2}{2}(2\frac{\mathrm{d}f}{\mathrm{d}u}) = \hbar^2\frac{\mathrm{d}}{\mathrm{d}u}(f)$. Therefore $[u,\, \hat{N}] = \mathrm{i}\hbar\hat{v}$. Similarly, $[\hat{v}, \hat{N}] = -\mathrm{i}\hbar u$. It is also standard to check that the "angular momentum variables" (x, y, z) satisfy : $[\hat{y},\, \hat{z}] = -\mathrm{i}\hbar\hat{x}$ and $[\hat{x},\, \hat{z}] = \mathrm{i}\hbar\hat{y}$.

Hence expression (4.4) equals

$$\hat{x}f \otimes (\mathrm{i}\hbar\hat{v})g + (\mathrm{i}\hbar\hat{y})f \otimes ug + \hat{y}f \otimes (-\mathrm{i}\hbar u)g + (-\mathrm{i}\hbar\hat{x})f \otimes \hat{v}g = 0.$$

The result follows. $\square$

Remark 4.3 Although the proof of Lemma 4.2 is interesting on its own, there is a theoretical reason for this lemma to be true, because our operators all derive from Weyl quantization of polynomial. And for such operators the following result is known: suppose that H_1 is a quadratic Hamiltonian and H_2 is any polynomial Hamiltonian function such that $\{H_1,\, H_2\} = 0$. Then Moyal's formula [26, 17, 12] yields, formally, $[\hat{H}_1,\, \hat{H}_2] = 0$. In our case J is quadratic in the variables $(u, v, x_1, x_2, \xi_1, \xi_2)$. This gives an alternative proof of Lemma 4.2. $\oslash$

4.3 Joint spectrum of $\hat{J},\, \hat{H}$

We have left to find the spectrum of $\hat{H}$ and of $\hat{J}$. First, we conjugate by the unitary transform in $\mathrm{L}^2(\mathbb{R}^2)$:

$$U : f(x_1, x_2) \to \sqrt{\hbar}f(\sqrt{\hbar}x_1, \sqrt{\hbar}x_2).$$

This has the effect of setting $\hbar = 1$ in the operator a_j :

$$Ua_jU^* = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_j} + x_j\right) =: A_j.$$

Next, it is convenient to use the Bargmann representation [2], which states that the operator A_j defined above and its adjoint A_j^* are unitarily equivalent to the operators $\frac{\partial}{\partial z_j}$ and z_j, respectively, acting on the Hilbert space of holomorphic functions on two variables $\mathrm{L}^2_{\mathrm{hol}}(\mathbb{C}^2,\, \pi^{-1}\,\mathrm{e}^{-|z|^2})$. (The notation z_j here is not exactly the same as the initial one in section 4.1, but we keep it for simplicity.)

The following lemma is standard.

Lemma 4.4 ([2])**.** *The function* $\frac{z_1^{\alpha_1}z_2^{\alpha}}{\sqrt{\alpha_1!\alpha_2!}} = \frac{z^\alpha}{\sqrt{\alpha!}}$*, where* $\alpha = (\alpha_1,\, \alpha_2)$*, is an eigenfunction of* $\hat{L}$ *with norm* 1 *and eigenvalue* $\hbar(\alpha_1 + \alpha_2 + 1)$.

Proof. The function $z_i^{\alpha_i}$ is an eigenfunction of $z_i\frac{\partial}{\partial z_i}$ with eigenvalue α_i. Since $\hat{L} = \hbar(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + 1)$, we get $\hat{L}(z^\alpha) = \hbar(\alpha_1 + \alpha_2 + 1)z^\alpha$.

We can compute $\|z^\alpha\|^2_{\mathrm{L}^2_{\mathrm{hol}}(\mathbb{C}^2,\, \pi^{-1}\,\mathrm{e}^{-|z|^2})} = \alpha!$. Therefore the function $\frac{z^\alpha}{\sqrt{\alpha!}}$ is a normalized eigenfunction of $\hat{L}$. $\square$

Next we find the eigenspace of $\hat{L}$ for the eigenvalue $\hbar(n + 1)$. Since the monomials $\{z^\alpha/\sqrt{\alpha!}\}_{\alpha\in\mathbb{N}^2}$ form a Hilbert basis of the Bargmann space, the space $\mathcal{H}_E = \ker(\hat{L} - \hbar(n + 1))$ is simply given by

$$\mathcal{H}_E = \mathrm{span}\{\frac{z^\alpha}{\sqrt{\alpha!}}\,|\,\alpha_1 + \alpha_2 = n\},$$

thus it is the space of homogeneous polynomials of degree n in $\mathbb{C}^2$. We will use for it the following basis :

$$\{z_2^n,\, z_1^n z_2^{n-1},\, \ldots,\, z_1^{n-1} z_2,\, z_1^n\}.$$

In order to understand the operator $\hat{H}$, we need to consider $\hat{z}$ and $\hat{N}$. The restriction of the operator $\hat{z} = \frac{\hbar}{2}(a_1 a_1^* - a_2 a_2^*)$ to the Hilbert space $\mathcal{H}_E$ in given in terms of this polynomial basis by $\hat{z}(z_1^k z_2^{n-k}) = \frac{\hbar}{2}(k-(n-k)) z_1^k z_2^{n-k}$. It follows that the matrix of $\hat{z} = \frac{\hbar}{2}(a_1 a_1^* - a_2 a_2^*)$ relative to this basis is the diagonal matrix

$$\frac{\hbar}{2}\begin{pmatrix} -n & 0 & \ldots & & & 0 \\ 0 & 2-n & 0 & & & 0 \\ 0 & 0 & 4-n & 0 & & 0 \\ & & & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & & 0 \\ 0 & 0 & \ldots & & 0 & n \end{pmatrix}$$

Notice that this shows that $\mathcal{H}_E$ is indeed invariant under the action of $\hat{z}$. Of course, a similar calculation can be done for $\hat{x}$ and $\hat{y}$ (see the proof of Proposition 4.7 below). Notice also that the eigenvalues of $\hat{z}$ range from $-\frac{\hbar}{2}n$ to $\frac{\hbar}{2}n$; in the case of the standard sphere S^2 (with $E = 2$), we have the relation $E = 2 = \hbar(n+1)$. Therefore the eigenvalues of $\hat{z}$ range from $-\frac{n}{n+1}$ to $\frac{n}{n+1}$. In the semiclassical limit $n \to \infty$, we recover the classical range $[-1, 1]$ of the hamiltonian z on S^2.

Next we consider the Bargmann representation for $\hat{N} = \frac{\hat{u}^2 + \hat{v}^2}{2}$. This time, we act of the Hilbert space $\mathrm{L}^2_{\mathrm{hol}}(\mathbb{C}_\tau,\, \pi^{-1}\mathrm{e}^{-|\tau|^2})$ and we obtain $\hat{N} = \hbar(\tau\frac{\partial}{\partial\tau} + \frac{1}{2})$.

The eigenfunctions of $\hat{N}$ are $\frac{\tau^\ell}{\sqrt{\ell!}}$ corresponding to the eigenvalue $\hbar(k + \frac{1}{2})$.

Lemma 4.5. *The spectrum of $\hat{J}$ is discrete, and we have*

$$\operatorname{spec}(\hat{J}) = \hbar\left(\frac{1-n}{2} + \mathbb{N}\right).$$

More precisely, for a fixed value $\lambda \in \hbar(\frac{1-n}{2} + \mathbb{N})$, let $\mathcal{E}_\lambda := \ker(\hat{J} - \lambda)$. Then

$$\mathcal{E}_\lambda = \operatorname{span}\left\{\tau^\ell \otimes z_1^k z_2^{n-k} \;\middle|\quad \hbar(\ell + \frac{1}{2} + k - \frac{n}{2}) = \lambda; \quad 0 \le k \le n; \quad \ell \ge 0\right\}.$$

In particular $\mathcal{E}_\lambda$ has dimension $1 + \min(n, \frac{\lambda}{\hbar} + \frac{n-1}{2})$.

Proof. In the double Bargmann representation, we have

$$\hat{J} = \operatorname{Id} \otimes (\hbar(\tau\frac{\partial}{\partial\tau} + \frac{1}{2})) + \frac{\hbar}{2}(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}) \otimes \operatorname{Id}.$$

Hence a simple computation gives

$$\hat{J}(\tau^\ell \otimes z_1^k z_2^{n-k}) = \hbar\left(\ell + \frac{1}{2} + k - \frac{n}{2}\right)(\tau^\ell \otimes z_1^k z_2^{n-k}) \tag{4.5}$$

so the corresponding eigenvalues are $\hbar(\ell + \frac{1}{2} + k - \frac{n}{2})$ where $0 \le k \le n$ and $n, \ell \ge 0$. This shows that $\hat{J}$ admits a complete set of eigenvectors. Hence $\ker(\hat{J} - \lambda)$ is spanned by the set of eigenvectors coming from this family and corresponding to the eigenvalue λ. This space is finite dimensional (hence $\hat{J}$ has discrete spectrum), and its dimension is the number of solutions (k, ℓ) to the equation $\hbar(\ell + \frac{1}{2} + k - \frac{n}{2}) = \lambda$ with constraints $0 \le k \le n;\ \ \ell \ge 0$, which is precisely $1 + \min(n, \frac{\lambda}{\hbar} + \frac{n-1}{2})$. ∎

The fact that $\mathcal{E}_\lambda$ is finite dimensional should be compared to the fact that the classical hamiltonian J is proper.

Corollary 4.6. *Given any $n \in \mathbb{N}$, and any $\lambda \in \hbar(\frac{1-n}{2} + \mathbb{N})$, the ordered set*

$$B_\lambda := \left\{ e_{\ell,k} := \frac{\tau^\ell}{\sqrt{\ell!}} \otimes \frac{z_1^k z_2^{n-k}}{\sqrt{k!(n-k)!}} \mid k = 0, 1, \ldots, \min(n, \frac{\lambda}{\hbar} + \frac{n}{2} - \frac{1}{2}), \text{ and } \ell = \frac{\lambda}{\hbar} + \frac{n}{2} - \frac{1}{2} - k \right\}.$$

is an orthonormal basis of $\mathcal{E}_\lambda$.

Our next goal is to compute the matrix of $\hat{H}$. More precisely, since $\hat{H}$ commutes with $\hat{J}$, the eigenspace $\mathcal{E}_\lambda$ is stable by $\hat{H}$. Thus, the spectral theory of $\hat{H}$ is merely reduced to the study of the restriction of $\hat{H}$ to $\mathcal{E}_\lambda$, which we explicitly compute below. Then the best way to depict the spectra of $\hat{J}$ and $\hat{H}$ is to display the *joint spectrum* (see figure 4.1), which is the set of $(\lambda, \nu) \in \mathbb{R}^2$ such that, for a common eigenfunction f, one has both

$$\hat{J}f = \lambda f \quad \text{and} \quad \hat{H}f = \nu f.$$

Let $\ell_0 := \frac{\lambda}{\hbar} + \frac{n}{2} - \frac{1}{2}$, $\mu = \min(\ell_0, n)$ and let

$$\beta_k := \sqrt{(\ell_0 + 1 - k)k(n - k + 1)}.$$

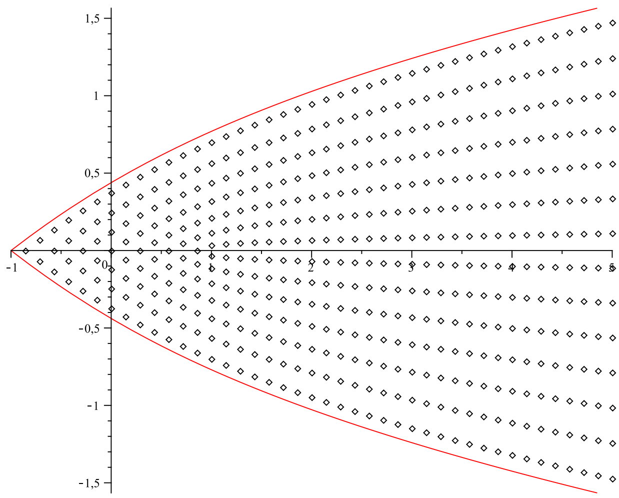

Figure 4.1: Semiclassical joint spectrum of $\hat{J}, \hat{H}$ and momentum map image juxtaposed, computed using a numerical diagonalization of the band matrix in Proposition 4.7. In all our computations we have chosen $E = 2$, which corresponds to the quantization of the standard sphere $x^2 + y^2 + z^2 = 1$. This implies the relation $2 = \hbar(n+1)$. Here $n = 13$, so $\hbar \simeq 1.14$.

Proposition 4.7. *The matrix $\mathrm{M}_{B_\lambda}(\hat{H})$ of the self-adjoint operator $\hat{H}$ on the basis B_λ is the symmetric matrix*

$$\mathrm{M}_{\mathcal{B}_\lambda}(\hat{H}) = \left(\frac{\hbar}{2}\right)^{\frac{3}{2}} \begin{pmatrix} 0 & \beta_1 & \ldots & & & 0 \\ \beta_1 & 0 & \beta_2 & & & 0 \\ 0 & \beta_2 & 0 & \beta_3 & & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & & & & & \beta_\mu \\ 0 & 0 & \ldots & & \beta_\mu & 0 \end{pmatrix}.$$

Proof. We start by evaluating $\hat{x}$ and $\hat{y}$ on this basis:

$$\begin{aligned}\hat{x}(z_1^k z_2^{n-k}) &= \frac{\hbar}{2}(k z_1^{k-1} z_2^{n-k+1} + (n-k) z_1^{k+1} z_2^{n-k-1})\\ \hat{y}(z_1^k z_2^{n-k}) &= \frac{\hbar}{2\mathrm{i}}(k z_1^{k-1} z_2^{n-k+1}) - (n-k) z_1^{k+1} z_2^{n-k-1})\end{aligned}$$

We introduce:

$$\alpha := \frac{1}{\sqrt{2\hbar}}(u + \hbar\frac{\partial h}{\partial u}), \qquad \alpha^* := \frac{1}{\sqrt{2\hbar}}(u - \hbar\frac{\partial h}{\partial u})$$

Hence $u(=\hat{u}) = (\alpha + \alpha^*)\sqrt{\frac{\hbar}{2}}$. Now we do the Bargmann representation

$$\hat{u} = \sqrt{\frac{\hbar}{2}}(\tau + \frac{\partial}{\partial \tau}), \qquad \hat{v} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial u} = \frac{(\alpha - \alpha^*)}{\mathrm{i}}\sqrt{\frac{\hbar}{2}} = \frac{1}{\mathrm{i}}\sqrt{\frac{\hbar}{2}}(\frac{\partial}{\partial \tau} - \tau).$$

Hence we obtain

$$\hat{u}(\tau^\ell) = \sqrt{\frac{\hbar}{2}}(\tau^{\ell+1} + \ell\tau^{\ell-1}), \qquad \hat{v}(\tau^\ell) = \frac{1}{\mathrm{i}}\sqrt{\frac{\hbar}{2}}(\ell\tau^{\ell-1} - \tau^{\ell+1}).$$

In what follows, for brevity of the notation, we write $c_k := z_1^k z_2^{n-k}$. Note that n is fixed. Recalling $\hat{H} = \frac{1}{2}(\hat{u}\otimes\hat{x} + \hat{v}\otimes\hat{y})$, we get

$$\begin{aligned}\hat{H}(\tau^\ell z_1^k z_2^{n-k}) &= \frac{1}{2}\Big(\Big(\frac{\hbar}{2}\Big)^{3/2}(\tau^{\ell+1} + \ell\tau^{\ell-1})(k c_{k-1} + (n-k)c_{k+1})\\ &- \Big(\frac{\hbar}{2}\Big)^{3/2}(\ell\tau^{\ell-1} - \tau^{\ell+1})(k c_{k-1} - (n-k)c_{k+1})\Big)\\ &= \frac{1}{2}\Big(\frac{\hbar}{2}\Big)^{3/2}\Big(k\tau^{\ell+1}c_{k-1} + \ell k\tau^{\ell-1}c_{k-1} + (n-k)\tau^{\ell+1}c_{k+1} + \ell(n-k)\tau^{\ell-1}c_{k+1}\\ &- \ell k\tau^{\ell-1}c_{k-1} + \ell(n-k)\tau^{\ell-1}c_{k+1} + k\tau^{\ell+1}c_{k-1} - (n-k)\tau^{\ell+1}c_{k+1}\Big)\\ &= \Big(\frac{\hbar}{2}\Big)^{3/2}(k\tau^{\ell+1}c_{k-1} + (n-k)\ell\tau^{\ell-1}c_{k+1}).\end{aligned} \tag{4.6}$$

Notice how this formula, together with Lemma 4.5, confirms that $\mathcal{E}_\lambda$ is stable under $\hat{H}$.

In order to have a better numerically prepared matrix (and a nicer-looking formula !), we next express everything in an orthonormal basis. Denote $e_{\ell,k} = \frac{\tau^\ell}{\sqrt{\ell!}}\frac{z_1^k z_2^{n-k}}{\sqrt{k!(n-k)!}}$ so that $e_{\ell,k}$ is an eigenvector of $\hat{J}$ of norm 1:

$$\begin{aligned}\hat{J}(e_{\ell,k}) &= \hbar(\ell + \frac{1}{2} + k - \frac{n}{2})e_{\ell,k} = \lambda e_{\ell,k}\\ \hat{H}(e_{\ell,k}) &= \Big(\frac{\hbar}{2}\Big)^{3/2}\frac{k\tau^{\ell+1}c_{k-1} + \ell(n-k)\tau^{\ell-1}c_{k+1}}{\sqrt{\ell!k!(n-k)!}}.\end{aligned} \tag{4.7}$$

On the other hand we have that $e_{\ell+1,k-1} = \frac{\tau^{\ell+1}c_{k-1}}{\sqrt{(\ell+1)!(k-1)!(n-k+1)!}}$ and that the first term of (4.7) is

$$\begin{aligned}\frac{k}{\sqrt{\ell!k!(n-k)!}}\tau^{\ell+1}c_{k-1} &= \frac{k}{\sqrt{\ell!k!(n-k)!}}\sqrt{(\ell+1)!(k-1)!(n-k+1)!}e_{\ell+1,k-1}\\ &= \sqrt{(\ell+1)k(n-k+1)}e_{\ell+1,k-1}.\end{aligned}$$

Similarly the second term of (4.7) is

$$\begin{aligned}\frac{\ell(n-k)\tau^{\ell-1}c_{k+1}}{\sqrt{\ell!k!(n-k)!}} &= \frac{\ell(n-k)}{\sqrt{\ell!k!(n-k)!}}\sqrt{(\ell-1)!(k+1)!(n-k-1)!}e_{\ell-1,k+1}\\ &= \sqrt{\ell(k+1)(n-k)}e_{\ell-1,k+1}.\end{aligned}$$

Since $\ell=\ell_0-k$, we get

$$\begin{aligned}\hat{H}(e_{\ell,k}) &= \left(\frac{\hbar}{2}\right)^{3/2}\left(\sqrt{(\ell_0-k+1)k(n-k-1)}e_{\ell+1,k-1}+\sqrt{(\ell_0-k)(k+1)(n-k)}e_{\ell-1,k+1}\right)\\ &= \left(\frac{\hbar}{2}\right)^{3/2}(\beta_k e_{\ell+1,k-1}+\beta_{k+1}e_{\ell-1,k+1}).\end{aligned}$$

This, of course, gives the statement of the proposition. □

4.4 The spectrum $\Sigma(n)$ of $\hat{H}|_{\ker(\hat{J}-\mathrm{Id})}$

In the next section, we will be particularly interested in the $\hat{J}$-eigenvalue $\lambda=1$, which corresponds to the J-critical value of the focus-focus point, in the classical system. Since $E=2=\hbar(n+1)$, we see that $\ell_0=\frac{n+1}{2}+\frac{n-1}{2}=n$. Therefore the dimension of $\ker(\hat{J}-\mathrm{Id})$ is equal to $n+1$. Notice that, for $\lambda<1$, the dimension of $\ker(\hat{J}-\lambda)$ is increasing linearly with slope 1 (with respect to the parameter k that we introduced above) whereas for $\lambda>1$ this dimension is constant, equal to $n+1$. This can be seen as a quantum manifestation of the Duistermaat-Heckmann formula [9].

5 Inverse spectral theory for quantum spin-oscillators

The theme of this section is to give evidence of the following conjecture being true in the case of coupled spin oscillators:

Conjecture 5.1. *A semitoric system is determined up to symplectic equivalence by its semiclassical joint spectrum (*i.e. *the set of points in $\mathbb{R}^2$ where on the x-axis we have the eigenvalues λ of $\hat{J}$, and on the vertical axes the eigenvalues of $\hat{H}$ restricted to the λ-eigenspace of $\hat{J}$). From any such spectrum one can construct explicitly the associated semitoric system.*

In this section we try to convey some ideas to explicitly compute all the symplectic invariants from the semiclassical spectrum. It might not necessarily be the optimal way to prove an inverse spectral result, as some quantities are more easily defined implicitly rather than explicitly by the spectrum. But we believe that, from a quantum viewpoint, having constructive formulas for the symplectic invariants is particularly valuable.

We emphasize the word "semiclassical" here : in order to recover the symplectic invariants we need be able to compute the joint spectrum for small values of $\hbar$. What can be said for a unique, fixed value of $\hbar$ is much harder question.

5.1 Polygon and height invariant

Recovering the polygon invariant is probably the easiest and most pictorial procedure, as long as one stays on a heuristic level. Making the heuristic rigorous should be possible along the lines of the toric case explained in [24] and [21], but we don't attempt to do it here.

The first thing to do is to recover the image of the classical moment map, including the position of the singular values. This could be done by a local examination of density of the joint eigenvalues.

Next, in order to recover the polygon invariant, we need to obtain the integral affine structure of the image of the momentum map. We know from [5, 24] that the joint spectrum possesses a semiclassical integral affine structure on the regular values of the momentum map. This integral affine structure can be extended to the elliptic boundaries, as explained in [24]. Thus, except along a vertical cut through the focus-focus critical value, one can develop this affine structure such that the joint eigenvalues become elements of the lattice $\hbar\mathbb{Z}^2$. See figure 5.1.

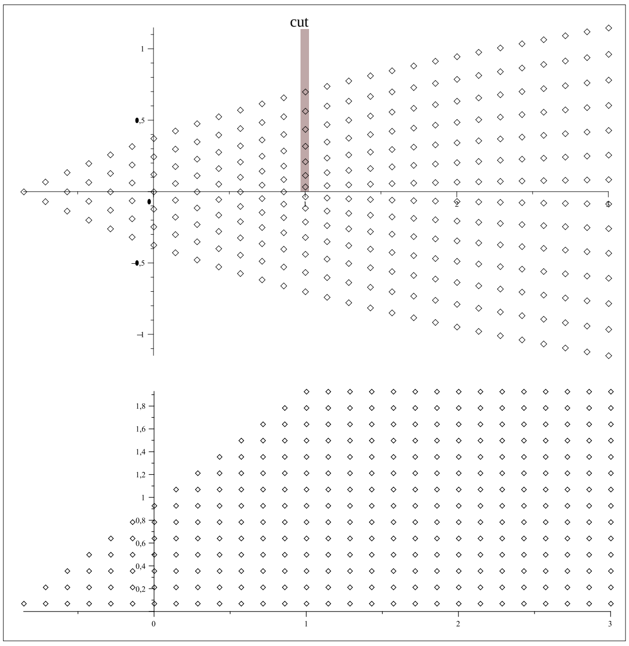

Figure 5.1: Recovering the polygon invariant. The top picture is the joint spectrum of $(\hat{J}, \hat{H})$. In the bottom picture, we have developed the joint eigenvalues into a regular lattice. One can easily check on this illustration that the number of eigenvalues in each vertical line in the same in both pictures.

The convex hull of the resulting set is a rational, convex polygonal set, depending on $\hbar$. Since the semiclassical affine structure is an $\hbar$-deformation of the classical affine structure, we see that, as $\hbar \to 0$, this polygonal set converges to the semitoric polygon invariant.

5.2 Semiclassical formula for the spectrum $\Sigma(n)$

In order to recover the Taylor series invariant from the spectrum, we need a precise description of this spectrum. There are two options : either describe the spectrum in regular regions, and then take the limit to the focus-focus critical value; or describe the spectrum directly in a small neighborhood of the focus-focus value. We choose the second option, because it seems more appropriate for a reasonably accurate numerical formula for the invariants, in the spirit of equation (2.28).

The drawback of this approach is that there is no result currently available giving the description of this spectrum. The singular Bohr-Sommerfeld rules of [22] would give the required result, in case $\hat{J}$ and $\hat{H}$ were pseudodifferential operators. Of course they are not, since the phase space $S^2 \times \mathbb{R}^2$ is not a cotangent bundle.

However they are semiclassical Toeplitz operators, in the sense of [4], and it is known that the algebra of Toeplitz operators is microlocally equivalent to the algebra of pseudodifferential operators [3]. Therefore, we propose the following conjecture.

Conjecture 5.2. *The formula in Corollary 6.8 in Vũ Ngọc's paper [22] holds also if the operators therein involved are Toeplitz instead of pseudodifferential.*

This conjecture may be stated in the following way. Let $\Sigma(n)$ be the spectrum of $\hat{H}|_{\ker(\hat{J}-\mathrm{Id})}$. For bounded $t \in \mathbb{R}$, the formula

$$\tilde{\lambda}(t) - \tilde{\epsilon}(t)\ln(2\hbar) - 2\arg\Gamma\Big(\frac{\mathrm{i}\tilde{\epsilon}(t)+1+j}{2}\Big) \in 2\pi\mathbb{Z} + \mathcal{O}(\hbar^\infty)$$

holds if and only if $\hbar t \in \Sigma(n) + \mathcal{O}(\hbar^\infty)$ with

(a) $\tilde{\lambda}(t) = \tilde{\lambda}(t;\hbar)$ admits an asymptotic expansion on integer ≥ -1 powers of $\hbar$ with smooth (=C^∞) coefficients in t starting with $\tilde{\lambda}(t) = \frac{1}{\hbar}\int_{\gamma_0}\alpha_0 + \mathrm{I}_{\gamma_0}(\tilde{\kappa}(t)) + \mu\frac{\pi}{2} + \mathcal{O}(\hbar)$.

(b) $\tilde{\epsilon}(t) = \tilde{\epsilon}(t;\hbar)$ has an asymptotic expansion on integer ≥ 0 powers of $\hbar$ with smooth coefficients in t starting with the second component of the vector $B(0,\, t) + \mathcal{O}(\hbar)$ where B is the 2×2 matrix such that $B(J'',\, H'')_m = (q_1,\, q_2)$.

(c) $I_{\gamma_0}(\tilde{\kappa}(t))$ is what is called the "principal value integral" of $\tilde{\kappa}(t)$, where $\tilde{\kappa}(t)$ is the 1-form on Λ_0 defined by

$$(\tilde{\kappa}(t)(\mathcal{X}_J),\, \tilde{\kappa}(t)(\mathcal{X}_H)) = (0,\, t) \iff (\tilde{\kappa}(t)(\mathcal{X}_{q_1}),\, \tilde{\kappa}(t)\mathcal{X}_{q_2})) = B(0,\, t) \tag{5.1}$$

Finally, $I_{\gamma_0}(\tilde{\kappa}^t)$ is defined in Proposition 6.15 of [22] as

$$I_{\gamma_0}(\tilde{\kappa}(t)) = \lim_{(s_1,\, s_2)\to(0,\,0)}\Big(\int_{A_0=\gamma_0(s_1)}^{B_0=\gamma_0(1-s_2)}\tilde{\kappa}(t) + \epsilon(t)\ln(r_{A_0}\rho_{B_0})\Big)$$

where $\epsilon(t)$ is the first order term of $\tilde{\epsilon}(t)$.

For a semitoric system, the matrix B is of the form $B = \begin{pmatrix} 1 & 0 \\ B_{21} & B_{22}\end{pmatrix}$, with $B_{22} \neq 0$. Thus we get

$$\epsilon(t) = B_{22}t.$$

Moreover, because of formula (5.1),

$$(\tilde{\kappa}(t)(\mathcal{X}_{q_1}),\, \tilde{\kappa}(t)(\mathcal{X}_{q_2})) = (0,\, B_{22}t).$$

Therefore we see that $\frac{\partial\tilde{\kappa}(t)}{\partial t} = B_{22}\kappa_{2,0}$, where $\kappa_{2,0}$ is the restriction to Λ_0 of the 1-form defined in equation (2.3). Thus, in view of equation (2.4), we get an explicit formula for the symplectic invariant a_2 :

$$a_2 = \frac{1}{B_{22}}\frac{\partial}{\partial t}\Big(I_{\gamma_0}(\tilde{\kappa}^t)\Big)\restriction_{t=0}. \tag{5.2}$$

Though we haven't worked it out here, a similar formula for the first invariant a_1 could be obtained along the same lines.

In the case of the coupled spin-oscillator, $B = \begin{pmatrix} 1 & 0 \\ 0 & 2\end{pmatrix}$, so $B_{22} = 2$ and $a_2 = \frac{1}{2}\frac{\partial}{\partial t}(I_{\gamma_0}(\tilde{\kappa}^t))\restriction_{t=0}$.

5.3 Obtaining a_2 from the spectrum $\Sigma(n)$

We show in this paragraph how the conjecture gives a way to obtain a_2. Using formula (5.2) above, an easy corollary of the conjecture is Theorem 7.6 in [22], which says that

$$\min\left(\frac{E_{k+1}-E_k}{\hbar}\right) = \frac{2\pi/B_{22}}{|\ln\hbar| + a_2 + \ln 2 + \gamma} + \mathcal{O}(\hbar) \tag{5.3}$$

for $\Sigma(n) = \{E_0 \leq E_1 \leq \ldots \leq E_n\}$. Here γ is Euler's constant.

From the spectrum we can calculate $t^{\min}(\hbar) = \min\left(\frac{E_{k+1}-E_k}{\hbar}\right)$ so

$$\frac{2\pi}{t^{\min}} = B_{22}(|\ln\hbar| + a_2 + \ln 2 + \gamma)(1+\mathcal{O}(\hbar)) = B_{22}(|\ln\hbar| + a_2 + \ln 2 + \gamma) + \mathcal{O}(\hbar\ln\hbar).$$

Therefore we may recover B_{22} as

$$B_{22} = \lim_{\hbar\to 0}\left(\frac{2\pi}{t^{\min}\,|\ln\hbar|}\right). \tag{5.4}$$

Because the convergence of this limit is very slow (of order $|\ln\hbar|^{-1}$), it is in practice much better to solve the system obtained with two different values of $\hbar$, which gives :

$$B_{22} = \frac{\frac{2\pi}{t^{\min}(\hbar_1)} - \frac{2\pi}{t^{\min}(\hbar_2)}}{\ln(\hbar_2/\hbar_1)} + \mathcal{O}(\hbar_1\ln\hbar_1) + \mathcal{O}(\hbar_2\ln\hbar_2). \tag{5.5}$$

Thus, if we choose $\hbar_2$ to be a fixed multiple of $\hbar = \hbar_1$, we get a convergence speed of order $\mathcal{O}(\hbar\ln\hbar)$, which is indeed much more reasonable.

Once B_{22} is known, it is easy to recover a_2, again through formula (5.3) :

$$a_2 = \lim_{\hbar\to 0}\left(\frac{2\pi}{B_{22}t^{\min}} - |\ln\hbar| - \ln 2 - \gamma\right), \tag{5.6}$$

and the convergence rate is again of order $\mathcal{O}(\hbar\ln\hbar)$.

5.4 Numerical approximation of a_2 using Maple

Using Proposition 4.7, we compute the spectrum $\Sigma(n)$ of the Spin-Oscillator example for various values of $n = 2/\hbar - 1$ by entering the matrix in the computer algebra system 'Maple' and ask for a numeric diagonalization. Then is it easy to implement the formulas (5.5) and (5.6).

From the general theory, the minimal eigenvalue spacing is obtained — at least in the limit $\hbar \to 0$, at the focus-focus critical value $H = 0$. This is confirmed from the numerics. In fact, using the recursion formula for the characteristic polynomial $D_n(X)$ of the matrix $\mathrm{M}_{B_\lambda}(\hat{H})$ (with $\ell_0 = n$) :

$$D_n(X) = XD_{n-1}(X) - \beta_n^2 D_{n-2}(X),$$

we prove by induction that $D_n(X)$ has the parity of $n+1$. In particular, the spectrum is symmetric : $\Sigma(n) = -\Sigma(n)$. When n is odd, 0 is not an eigenvalue ($D_n(0) = (-1)^{(n-1)/2}\beta_1\beta_3\cdots\beta_n$), and hence the smallest spacing is simply twice the smallest positive eigenvalue :

$$t^{\min}(\hbar) = 2E_{[\frac{n}{2}]+2}/\hbar \quad \text{with } \hbar = \frac{2}{n+1}.$$

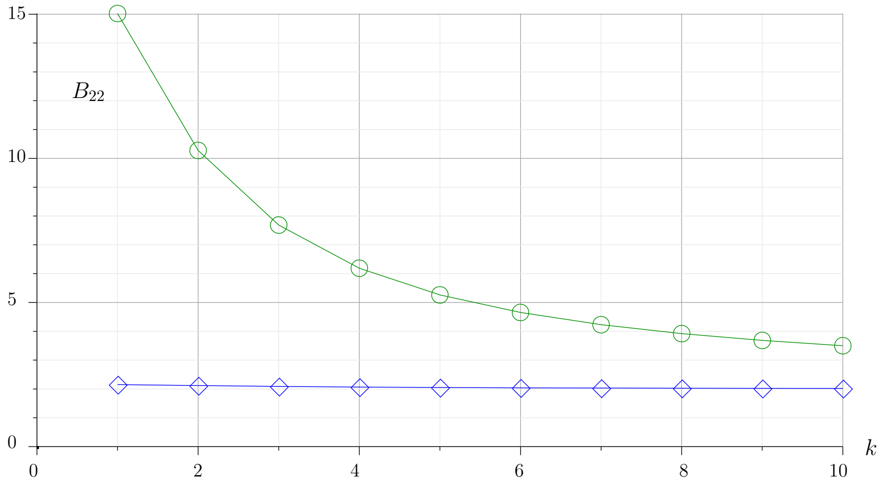

Figure 5.2: Recovering the coefficient B_{22} (which is equal to 2 in our example). The horizontal scale is logarithmic: the integer abscissa k corresponds to $n = 2^k + 1$. Thus $\hbar$ starts at 0.5 and decreases to the right to reach $1/513 \simeq 0.002$. The top curve — with circles — is the result of formula (5.4), which indeed converges very slowly. The curve with diamonds is obtained by the accelerated formula (5.5).

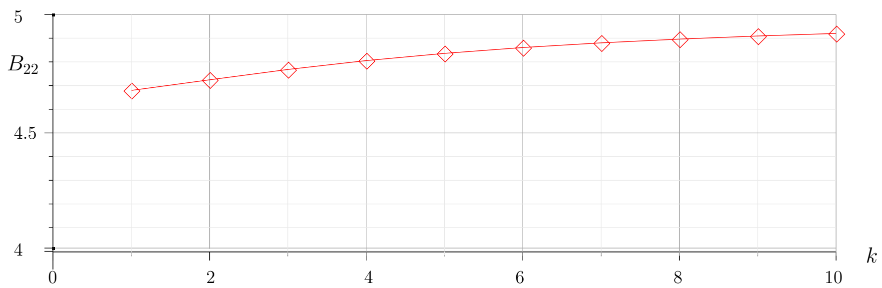

Figure 5.3: Recovering the invariant a_2. The graph plots the values of $a_2/\ln 2$ (which should be 5 in our example) computed using the formula (5.6). The horizontal scale is the same is in figure 5.2.

The results of our numerical experiments are plotted in figures 5.2 and 5.3. They should be compared to the theoretical values of Theorem 2.14.

References

[1] M. F. Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14(1):1–15, 1982.

[2] V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform I. *Comm. Pure Appl. Math.*, 19:187–214, 1961.

[3] L. Boutet de Monvel and V. Guillemin. *The spectral theory of Toeplitz operators*. Number 99 in Annals of Mathematics Studies. Princeton university press, 1981.

[4] L. Charles. Berezin-toeplitz operators, a semi-classical approach. *Commun. Math. Phys.*, 239(1-2):1–28, 2003.

[5] R. Cushman and J. J. Duistermaat. The quantum spherical pendulum. *Bull. Amer. Math. Soc. (N.S.)*, 19:475–479, 1988.

[6] T. Delzant. Hamiltoniens périodiques et image convexe de l'application moment. *Bull. Soc. Math. France*, 116:315–339, 1988.

[7] M. Dimassi and J. Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.

[8] J. J. Duistermaat. On global action-angle variables. *Comm. Pure Appl. Math.*, 33:687–706, 1980.

[9] J. J. Duistermaat and G. J. Heckman. On the variation in the cohomology of the symplectic form of the reduced phase space. *Invent. Math.*, 69:259–268, 1982.

[10] L.H. Eliasson. *Hamiltonian systems with Poisson commuting integrals*. PhD thesis, University of Stockholm, 1984.

[11] M. Garay and D. van Straten. Classical and quantum integrability. preprint ArXiv 0802.1647, 2008.

[12] H. J. Groenewold. On the principles of elementary quantum mechanics. *Physica*, 12:405–460, 1946.

[13] M. Gross and B. Siebert. Mirror symmetry via logarithmic degeneration data. I. *J. Differential Geom.*, 72(2):169–338, 2006.

[14] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, 67(3):491–513, 1982.

[15] B. Kostant and Á. Pelayo. Introduction to Geometric Quantization. Monograph to appear in Springer-Verlag.

[16] N. C. Leung and M. Symington. Almost toric symplectic four-manifolds. Preprint `math.SG/0312165`, 2003.

[17] J. E. Moyal. Quantum mechanics as a statistical theory. *Proc. Cambridge Philos. Soc.*, 45:99–124, 1949.

[18] Á. Pelayo and S. Vũ Ngọc. Semitoric integrable systems on symplectic 4-manifolds. *Invent. Math.*, 177(3):571–597, 2009.

[19] Á. Pelayo and S. Vũ Ngọc. Constructing integrable systems of semitoric type. *Acta Math.*, 2010. (to appear).

[20] M. Symington. Four dimensions from two in symplectic topology. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 153–208. Amer. Math. Soc., Providence, RI, 2003.

[21] S. Vũ Ngọc. Symplectic inverse spectral theory for pseudodifferential operators. HAL preprint, June 2008. To appear in a Volume dedicated to Hans Duistermaat.

[22] S. Vũ Ngọc. Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type. *Comm. Pure Appl. Math.*, 53(2):143–217, 2000.

[23] S. Vũ Ngọc. On semi-global invariants for focus-focus singularities. *Topology*, 42(2):365–380, 2003.

[24] S. Vũ Ngọc. *Systèmes intégrables semi-classiques: du local au global*. Number 22 in Panoramas et Syhthèses. SMF, 2006.

[25] S. Vũ Ngọc. Moment polytopes for symplectic manifolds with monodromy. *Adv. in Math.*, 208:909–934, 2007.

[26] H. Weyl. *The theory of groups and quantum mechanics*. Dover, 1950. Translated from the (second) German edition.

[27] J. Williamson. On the algebraic problem concerning the normal form of linear dynamical systems. *Amer. J. Math.*, 58(1):141–163, 1936.

[28] Nguyên Tiên Zung. A topological classification of integrable hamiltonian systems. In R. Brouzet, editor, *Séminaire Gaston Darboux de géometrie et topologie différentielle*, pages 43–54. Université Montpellier II, 1994-1995.

[29] Nguyên Tiên Zung. Symplectic topology of integrable hamiltonian systems, I: Arnold-Liouville with singularities. *Compositio Math.*, 101:179–215, 1996.

Alvaro Pelayo
University of California–Berkeley
Mathematics Department
970 Evans Hall # 3840
Berkeley, CA 94720-3840, USA.
E-mail: apelayo@math.berkeley.edu

Vũ Ngọc San
Institut de Recherches Mathématiques de Rennes
Université de Rennes 1
Campus de Beaulieu
35042 Rennes cedex (France)
E-mail: san.vu-ngoc@univ-rennes1.fr